ROM's of ROM's make sense: semi- analytical approach to simplified reduced order models, inertial manifolds and global bifurcations in BWR dynamics

Roberto Suárez-Ántola

Departamento de Matemática – Facultad de Ingeniería y Tecnologías
Universidad Católicadel Uruguay, Montevideo, Uruguay
rsuarez@ucu.edu.uy

Dirección General de Secretaría
Ministerio de Industria, Energía y Minería, Montevideo, Uruguay
roberto.suarez@miem.gub.uy

ABSTRACT

One of the goals of nuclear power systems design and operation is to restrict the possible states of certain critical sub-systems to remain inside a certain bounded set of admissible states and state variations.
In the framework of an analytic or numerical modeling process of a BWR power plant, this could imply first to find a suitable approximation to the solution manifold of the system of nonlinear partial differential equations describing the stability behavior, and then a classification of the different solution types concerning their relation with the operational safety of the power plant. Inertial manifold theory gives a foundation for the construction and use of reduced order models (ROM's) of reactor dynamics to discover and characterize meaningful bifurcations that may pass unnoticed during digital simulations done with full scale computer codes of the nuclear power plant. The main aspects of approximate inertial manifolds and forms are briefly reviewed in the introduction of the paper.A complete numerical study of reactor dynamics using a realistic ROM currently involves the digital simulation of the behavior of approximately twenty state variables interrelated by a corresponding system of coupled nonlinear ordinary differential equations. The success of hybrid analytical-numerical bifurcation codes to detect interesting behavior, such as global bifurcations in BWR's, may be enhanced by studying suitable simplifications of ROM's, that is ROM's of ROM's. A previous generalization of the classical March-Leuba's model of BWR is briefly reviewed and a nonlinear integral-differential equation in the logarithmic power is derived. The asymptotic method developed by Krilov, Bogoliubov and Mitropolsky (KBM) is applied to obtain approximate equations of evolution for the amplitude and the phase of a manifold of oscillatory solutions jointly with a relation between an offset and the abovementioned amplitude. First, to exemplify the method working with a simpler problem, the KBM tentative solution (ansatz) is applied to construct approximate solutions of, and to study local bifurcations in, a van der Pol equation with continuous and discrete distribution of time delays. Then, the afore-mentioned ansatz is applied to the full nonlinear integral-differential equation of the BWR model. Analytical formulae are derived for the offset, the rate of change in the phase (the instantaneous frequency of oscillation) and the rate of change in the amplitude of oscillation, given as functions of the amplitude and the model parameters (steady state power and coolant flow, temperature and void reactivity coefficients, fuel to coolant heat transfer coefficient and other parameters from neutronics and thermal hydraulics). The obtained analytical formulae are applied to start a semi-analytical, mainly qualitative, approach to bifurcations and stability of the steady states located in different regions of parameters space. This includes a qualitative discussion of the possibility of both, super and subcritical Poincaré-Andronov-Hopf bifurcations, as well as a Bautin's bifurcation scenario. The preliminary qualitative results outlined in this study are consistent with results of recent digital simulations done with a full-scale reduced order model of BWR (PSI-TU Valencia-TU Dresden) and with the results obtained with the application of hybrid approaches to bifurcation theory done with the simplified March-Leuba's model of BWR.

Key words: **nuclear reactor dynamics, differential-integral equations, reduced order models of boiling water reactors, averaging methods, Hopf supercritical and subcritical bifurcation, Bautin scenario, nuclear reactor stability and control.**

1. INTRODUCTION

One of the initial goals of nuclear power systems design and operation remains of practical importance: to impose suitable restrictions so that the possible states of certain critical subsystems, during steady operation and during transients, always remain suitably bounded. Due to power increase which is the current trend, stability becomes a limit design concern. The introduction of high efficiency fuels enables operation at higher power densities resulting in higher void feedback reactivities and decreased heat transfer time constants. Both effects tend to lower the stability of the operating points in Boiling Water Reactors (BWR).
In phase power oscillations have been a recurrent research subject since the construction and operation of the first generation of BWR up to the present time. In fact, most instability events in commercial nuclear power plants with BWR appeared during stability tests. However, a significant fraction of the total has occurred during regular operation (near 25%).

Global sources of oscillation are related with neutron kinetic-thermal-hydraulic instabilities that appear under certain circumstances due to physical interactions that span the whole core.
The subject of the present paper is the mathematical analysis of in phase direct-loop power oscillations ([1]) using asymptotic methods.

Full scale realistic mathematical modeling must be done beginning with a suitable coupled system of non-linear partial differential equations. Then, the corresponding computational code for digital simulation of steady states and transients is constructed and tested.
The field equations are dissipative nonlinear partial differential equations (DNPDEs) [2].
The long-time evolution of the solutions of certain DNPDEs appear to be equivalent to the solutions of certain ordinary differential equations (ODEs), because asymptotic evolution of the solutions of DPDE occurs near and exponentially approaching to a finite dimensional manifold, the so-called inertial manifold [2], [3].
An inertial manifold of solutions of a partial differential equation is a finite dimensional, compact and invariant set that attracts exponentially all trajectories that begin in a certain neighborhood of the manifold and is asymptotically complete in the following sense: for each trajectory that begins in a neighborhood of the inertial manifold, there is a trajectory included in the inertial manifold such that the distance between these trajectories tends to zero exponentially [2], [3].
Let us review some of the main ideas related with inertial manifolds and inertial forms [3]. We begin with a system of nonlinear partial differential equations in an infinite dimension Hilbert space H whose points x represent possible states of the corresponding dynamic system:

$$\frac{dx}{dt} = \hat{A}_\gamma(x) + \hat{N}_\gamma(x) + f_\gamma \qquad \text{(i1)}$$

The components of the state vector can be fields (like the neutron flux, the distribution of temperatures in fuel, moderator, coolant and structural elements of the nuclear reactor core,

coolant velocities, coolant void fractions) and time functions (to describe the state of components that have effects on nuclear reactor dynamics but are not in the core).

The term f_γ is composed by external fields or forcing functions of time. It is considered as given. In the nuclear reactor case it comprises control rods and other elements of the control system, as well as certain mechanical or electric loads in case of a power reactor.

Here γ represents a set of points in parameter's space Γ. Each point consists of a possible set of values of these parameters. So, for each point $\gamma \in \Gamma$ there is a dynamical system given by (i1). All these dynamic systems have the same Hilbert space H of possible states. In the nuclear reactor case the fields are defined in a bounded domain (the extrapolated core) and the boundary conditions are taken into account when the operators and the state space are defined.

The linear operator $\hat{A}_\gamma$ has adjoint operator. The nonlinear operator $\hat{N}_\gamma$ comprises all the nonlinearities in the dynamic system. When it exists, the inertial manifold can be determined by a graph in Hilbert space. To construct this graph, the eigenvalues and eigenvectors of the linear operator in (i1) must be known.

We suppose here that the linear operator $\hat{A}_\gamma$ has a set of simple eigenvalues $\sigma_j(\gamma)$ with their corresponding eigenvectors v_j:

$$\hat{A}_\gamma(v_j) = \sigma_j(\gamma)\cdot v_j \quad j = 1,2,3,... \qquad \text{(i2)}$$

To simplify we consider that the operator has negative eigenvalues ordered in a decreasing sequence: $0 > \sigma_1(\gamma) > \sigma_2(\gamma) > ...$ To assure the existence of a graph Φ in the space H that can be a candidate to inertial manifold, it is necessary that the spectrum of $\hat{A}_\gamma$ has a large gap between a certain eigenvalue $\sigma_m(\gamma)$ and the next one. In terms of absolute values:

$$|\sigma_{m+1}(\gamma)| - |\sigma_m(\gamma)| \geq c(\gamma)\cdot\left(\sqrt{|\sigma_{m+1}(\gamma)|} + \sqrt{|\sigma_m(\gamma)|}\right) \qquad \text{(i3)}$$

The positive constant $c(\gamma)$ depends of the linear operator $\hat{A}_\gamma$.

Now, we assume this gap condition and consider the first m eigenvalues, their corresponding eigenvectors $v_1, v_2, ..., v_m$ and the finite dimension linear subspace E_s generated by these eigenvectors: $\{\sigma_1(\gamma), \sigma_2(\gamma), ..., \sigma_m(\gamma)\}$ (i3) $\{v_1, v_2, ..., v_m\} \to E_s$ (i4)

To the remaining set of eigenvalues $\{\sigma_{m+1}(\gamma), \sigma_{m+2}(\gamma), ...\}$ corresponds an infinite set of eigenvectors $v_{m+1}, v_{m+2}, ...$ that spans an infinite dimension linear subspace E_f:

$\{v_{m+1}, v_{m+2}, ...\} \to E_f$ (i5) The direct sum of these subspaces is the Hilbert space of the problem: $H = E_s \oplus E_f$ (i6) $E_s \cap E_f = \{\phi\}$ (i7) (As usual ϕ is the void set).

Then, it is possible to decompose the state vector in the sum of a vector p of slow variables and a vector q of fast variables: $x = p + q \leftrightarrow \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}$ (i8) Vectors $\tilde{p}$ and $\tilde{q}$ in (i8) are the projections of the state vector onto the subspaces E_s and E_f, respectively:

$$\tilde{p} = \hat{P}_\gamma(x) \in E_s \qquad \text{(i9)} \qquad \tilde{q} = \hat{Q}_\gamma(x) \in E_f \qquad \text{(i10)}$$

The projection operator $\hat{P}_\gamma$ projects to the slow subspace, and $\hat{Q}_\gamma$ projects to the fast subspace. In self-explicative terms: $p \leftrightarrow \begin{bmatrix} \tilde{p} \\ \tilde{0} \end{bmatrix}$ (i11a) $q \leftrightarrow \begin{bmatrix} \tilde{0} \\ \tilde{q} \end{bmatrix}$ (i11b)

Then the graph $\Phi_{IM,\gamma}$ in Hilbert space that defines the inertial manifold is given by a function from $\hat{P}_\gamma H$ to $\hat{Q}_\gamma H$ that can be written: $q = \Phi_{IM,\gamma}(p)$ (i12)

If we consider two points x_1 and x_2 in the inertial manifold included in the Hilbert space of the problem, and make the decomposition $x_1 = p_1 + q_1$, $x_2 = p_2 + q_2$ then if the graph function verifies $\|q_2 - q_1\| = \|\Phi_{IM,\gamma}(p_2) - \Phi_{IM,\gamma}(p_1)\| \le K_\gamma \|p_2 - p_1\|$ for some positive constant K_γ, the asymptotic behavior of two trajectories in the exact inertial manifold is given

$$\lim_{t\to\infty} \|q_2 - q_1\| = \lim_{t\to\infty} \|\Phi_{IM,\gamma}(p_2) - \Phi_{IM,\gamma}(p_1)\| \le K_\gamma \lim_{t\to\infty} \|p_2 - p_1\|$$

The construction of the exact inertial manifold represented by formula (i12) is in general not attainable, so an approximate inertial manifold is constructed for most of the problems. Let us briefly see this. We make the following decomposition of the linear operator:

$$\hat{A}_\gamma \leftrightarrow \begin{bmatrix} \hat{A}_{\gamma,s} & \hat{0} \\ \hat{0} & \hat{A}_{\gamma,f} \end{bmatrix} \quad \text{(i13)} \qquad \text{Then it is possible to write:}$$

$$\hat{A}_\gamma(x) \leftrightarrow \begin{bmatrix} \hat{A}_{\gamma,s} & \hat{0} \\ \hat{0} & \hat{A}_{\gamma,f} \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \hat{A}_{\gamma,s}(\tilde{p}) \\ \hat{A}_{\gamma,f}(\tilde{q}) \end{bmatrix} \quad \text{(i14)}$$

Here by definition: $\tilde{P}_\gamma \hat{A}_\gamma = \hat{A}_{\gamma,s}$ (i15a) $\tilde{Q}_\gamma \hat{A}_\gamma = \hat{A}_{\gamma,f}$ (i15b)

Then, the equations for the slow variables coupled with the fast variable are the following:

$$\frac{d\tilde{p}}{dt} = \hat{A}_{\gamma,s}(\tilde{p}) + \hat{P}_\gamma \hat{N}_\gamma\left(\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}\right) + \hat{P}_\gamma f_\gamma \quad \text{(i16a)}$$

And the equations for the fast variables coupled with the slow variables are:

$$\frac{d\tilde{q}}{dt} = \hat{A}_{\gamma,f}(\tilde{q}) + \hat{Q}_\gamma \hat{N}_\gamma\left(\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}\right) + \hat{Q}_\gamma f_\gamma \quad \text{(i16b)}$$

As the state variables in (i16b) are fast relative to the state variables in (i16a), let us consider the first ones relaxed to equilibrium with the slow ones:

$$\hat{A}_{\gamma,f}(\tilde{q}) + \hat{Q}_\gamma \hat{N}_\gamma\left(\begin{bmatrix} \tilde{p} \\ \tilde{\Phi}_{AIM,\gamma}(\tilde{p}) \end{bmatrix}\right) + \hat{Q}_\gamma f_\gamma \approx 0 \quad \text{(i17)}$$

From (i17) we obtain an implicit equation for an approximate inertial manifold $\tilde{q} = \tilde{\Phi}_{AIM,\gamma}(\tilde{p})$: $\tilde{q} = \tilde{\Phi}_{AIM,\gamma}(\tilde{p}) = \hat{A}_{\gamma,f}^{-1}\left(\hat{Q}_\gamma f_\gamma + \hat{Q}_\gamma \hat{N}_\gamma\left(\begin{bmatrix} \tilde{p} \\ \tilde{\Phi}_{AIM,\gamma}(\tilde{p}) \end{bmatrix}\right)\right)$ (i18)

This last implicit equation can be simplified to this explicit one:

$$\tilde{q} = \tilde{\Phi}_{AIM,\gamma}(\tilde{p}) \approx \hat{A}_{\gamma,f}^{-1}\left(\hat{Q}_\gamma f_\gamma + \hat{Q}_\gamma \hat{N}_\gamma\left(\begin{bmatrix} \tilde{p} \\ \tilde{0} \end{bmatrix}\right)\right) \quad \text{(i19)}$$

In any case, substituting $\widetilde{q} = \widetilde{\Phi}_{AIM,\gamma}(\widetilde{p})$ in (i15a) we obtain an approximate inertial form:

$$\frac{d\widetilde{p}}{dt} \approx \hat{A}_{\mu}(\widetilde{p}) + \hat{P}_{\mu}\,\hat{N}_{\mu}\left(\begin{bmatrix} \widetilde{p} \\ \widetilde{\Phi}_{AIM,\gamma}(\widetilde{p}) \end{bmatrix}\right) + \hat{P}_{\mu} f \tag{i20}$$

From (i8), (i11a), (i11b) and (i19) we can construct the approximate inertial manifold in the original Hilbert space of the problem:

$$q = \Phi_{AIM,\gamma}(p) \leftrightarrow \begin{bmatrix} \widetilde{0} \\ \widetilde{q} \end{bmatrix} = \begin{bmatrix} \widetilde{0} \\ \widetilde{\Phi}_{AIM,\gamma}(\widetilde{p}) \end{bmatrix} \approx \begin{bmatrix} \widetilde{0} \\ \hat{A}^{-1}{}_{\gamma,f}\left(\hat{Q}_{\gamma} f_{\gamma} + \hat{Q}_{\gamma}\,\hat{N}_{\gamma}\left(\begin{bmatrix} \widetilde{p} \\ \widetilde{0} \end{bmatrix}\right)\right) \end{bmatrix} \tag{i21}$$

The above reviewed ideas, extended to the case in which the operator has complex conjugate eigenvalues, constitute a rigorous mathematical background that justifies the introduction of reduced order models (ROMs) in nuclear reactor dynamics and control: most often, after a fast initial stage, the transients are slow (and in certain cases, delocalized enough) to allow the use of ODEs instead of DNPDEs in stability studies.

However, a rigorous reduction of a complex field model of nuclear reactor dynamics to its approximate inertial form in its approximate inertial manifold is seldom done.

Local bifurcations and stability are studied from ROM's equations that are obtained mainly ab-initio, making suitable hypothesis in order to directly construct mathematical models like the ones to be considered below.

Centre manifold theory ([2]) can be applied to further reduce the dimensionality of these models in the neighbourhood of a point in parameter space where the real parts of a pair of complex conjugate eigenvalues vanish. (By the way, the inertial manifold is central in this case.)

The use of ROMs in nonlinear stability analysis of BWR allows the application of a suitable combination of semi-analytical bifurcation theory with digital simulation of the dynamics using well studied software for ordinary differential equations (see reference [4]). The results thus obtained can be used to guide the numerical simulation using full system codes.

A typical ROM for a BWR, such as the very effective one developed in Paul Scherrer Institute, Valencia Technical University and Dresden Technical University, may have a number of state variables near 20, interrelated by a corresponding system of near 20 nonlinear ordinary differential equations [5]. However, the classical March-Leuba's simplified model of a BWR reviewed in reference [6] together with other lumped parameter models of BWR behavior, has at most 5 state variables, but reproduces some of the main characteristic of reactor's dynamics and is easier to study. When a simplified model can be derived from a more comprehensive ROM, it can be considered as a ROM of a ROM. ROM's of ROM's make sense because when the mathematical model of the nuclear reactor is simple enough, it is sometimes possible to study local bifurcations of steady states and limit cycles and even some global bifurcations, using approximate analytical methods. The results thus obtained usually offer a useful guide to deeper stability and bifurcation studies using the same model or more complex ROM's of the same physical system.

In a previous work, March-Leuba's classical ROM for coupled thermo-hydraulic-neutronic instabilities was re-visited and the excess void reactivity that appears in the model was generalized, although retaining its simplicity [7]. A single nonlinear integral-differential equation for power kinetics with feedback in terms of logarithmic power, equivalent to the system of equations of the generalized classical model, was obtained. Then, the possibility of studying the onset of power oscillations in the generalized model applying the asymptotic method developed by Krilov, Bogoliubov and Mitropolsky (KBM) to the above mentioned nonlinear integral-differential equation was considered [7].
However, as will be seen below, the ansatz (tentative solution) employed in reference [7] to apply KBM **is an acceptable approximation only for small (relative to 1) dimensionless amplitudes of oscillation of the reactor´s logarithmic power**.

The purposes of this paper are:
(1) Review the above-mentioned generalization of March-Leuba's classical ROM for coupled thermo-hydraulic-neutronic instabilities, as well as the derivation of the mentioned integral-differential equation for power kinetics with feedback in terms of logarithmic power.
(2) Introduce, in the ansatz for the logarithmic power, an offset, assumed to be a function of the amplitude of the oscillations in logarithmic power, and determine this function using a method due to Denn and Black [8], before applying KBM as such.
(3) Develop a zero order KBM analysis to obtain equations for the amplitude and phase of oscillations as well-defined analytical functions of the parameters of the mathematical model. Even a zero-order approximation could help in the analysis of the possibility of the onset of power oscillations under appropriate circumstances.

2. A DIRECT LOOP MODEL FOR BWR

We begin with point kinetics with feedback in terms of the reactor power $P(t)$ [8]:

$$\frac{dP}{dt} = \frac{(\rho - \beta)}{\Lambda} \cdot P + \frac{\beta}{\Lambda} \cdot \int_0^\infty D(u) \cdot P(t-u) \cdot du \qquad (1)$$

$D(t)$ is the delayed neutrons kernel: $D(t) = \sum_{j=1}^{I} \frac{\beta_j}{\beta} \cdot \lambda_j \cdot e^{-\lambda_j \cdot t} \qquad \sum_j \beta_j = \beta \qquad \int_0^\infty D(u) \cdot du = 1$

Λ is the mean generation time for the reactor and β is the total fraction of delayed neutrons.

2.1 Equations for March-Leuba's reduced order model of a BWR and a generalization of void feedback reactivity

This model has one group of delayed neutron emitters of concentration $c(t)$: in this case $D(t) = \lambda \cdot e^{-\lambda \cdot t}$ Thermal power $P(t)$, fuel temperature $T(t)$ and void reactivity $\rho_\alpha(t)$ are used (besides $c(t)$) as state variables interrelated by a system of ordinary differential equations.
As our intention is to work with a single integral-differential equation in terms of reactor power, we eliminate $c(t)$ and begin directly with Eq. (1).

Let us suppose that there is a reference steady state: $P = P_0 \quad T = T_0 \quad \rho_\alpha = 0$ (2)

Then, reactivity and feedback variables for March-Leuba´s simplified model of a BWR are given by the following equations [9]:

$$\rho = \rho_\alpha + \alpha_T \cdot (T - T_0) \tag{3a}$$

$$C \cdot \frac{d(T - T_0)}{dt} = (P - P_0) - h \cdot (T - T_0) \tag{3b}$$

$$\frac{d^2\rho_\alpha}{dt^2} + b_1 \cdot \frac{d\rho_\alpha}{dt} + b_2 \cdot \rho_\alpha = b_3 \cdot (T - T_0) + b_4 \cdot \frac{d(T - T_0)}{dt} \tag{3c}$$

Here ρ represents the excess total reactivity, ρ_α is the excess void reactivity, α_T is a Doppler temperature reactivity coefficient (α_T is always **negative**), $T - T_0$ is the excess fuel temperature relative to steady state conditions, $P - P_0$ is excess power relative to the steady state, C is fuel heat capacity, and h is a fuel-coolant heat transport coefficient.

The coefficients in the differential equation that relate excess void reactivity with excess fuel temperature b_1 b_2 b_3 and b_4 are in principle adjustable empirical parameters. Both b_1 and b_2 are found to be positive, while b_3 and b_4 are found to be negative.

If we make $b_4 = 0$, we obtain a particular case. This case is very well studied, both from the digital simulation standpoint and from a semi-analytical bifurcation theory viewpoint [10].

Taking Laplace's transform on both sides of Eq. (3c), and representing Laplace's variable by s, the transference function from temperature to void reactivity is obtained:

$$\overline{H}(s) = \frac{b_3 + b_4 \cdot s}{s^2 + b_1 \cdot s + b_2} \tag{4}$$

Introducing the **negative** static coefficient $\alpha_V = \frac{b_3}{b_2}$ to relate excess void reactivity with excess fuel temperature and assuming zero initial conditions for $t = -\infty$, the dynamic behavior of the excess void reactivity as a function of the excess fuel temperature $\Delta T(t) = T(t) - T_0$ can be written thus:

$$\rho_\alpha = \alpha_V \cdot \int_0^\infty h_\alpha(u) \cdot \Delta T(t - u) \cdot du \tag{5}$$

Introducing a damping factor ζ_ρ and a natural angular frequency ω_ρ by the relations $b_1 = 2.\zeta_\rho.\omega_\rho$ and $b_2 = \omega_\rho^{\,2}$, for $\zeta_\rho < 1$ the impulse response function $h_\alpha(t)$ is given by:

$$h_\alpha(t) = \omega_\rho \cdot e^{-\zeta_\rho \cdot \omega_\rho \cdot t} \cdot \left[\frac{\left(1 - \frac{\zeta_p \cdot \omega_p \cdot b_4}{b_3}\right)}{\sqrt{1 - \zeta_\rho^{\,2}}} \cdot \sin\left(\omega_\rho \cdot \sqrt{1 - \zeta_\rho^{\,2}} \cdot t\right) + \omega_p \cdot \left(\frac{b_4}{b_3}\right) \cdot \cos\left(\omega_\rho \cdot \sqrt{1 - \zeta_\rho^{\,2}} \cdot t\right) \right] \tag{6}$$

Now, as shown in the appendix of reference [1], it is possible to derive the following approximate formulae for the parameters of the transfer function given by Eq. (4):

$$b_1 \approx \frac{6}{\tau} \quad (7a) \qquad b_2 \approx \frac{12}{\tau^2} \quad (7b) \qquad b_3 \approx -\frac{6 \cdot \Theta \cdot H^2}{\tau^2} \quad (7c) \qquad b_4 \approx -\frac{\Theta \cdot H^2}{\tau} \quad (7d)$$

Here $\tau \approx \frac{H}{\overline{V_0}}$ is **the residence time of steam bubbles** in the channel, H is channel height, Θ is a positive parameter, and $\overline{V_0}$ is a mean void propagation velocity. Then, from formulae (7a) and (7b) it follows: $\omega_\rho = \sqrt{b_2} = \frac{\sqrt{12}}{\tau}$ and $\zeta_\rho = \frac{b_1}{2 \cdot \omega_p} = \frac{3}{\sqrt{12}} < 1$

As shown in reference [1], the derivation of the equations (7) for b_1 b_2 b_3 and b_4 in the case of the direct loop mechanism, is done substituting the exponential $\exp[-s \cdot \tau]$ by its Padé's (2, 2) approximation.

However, **if a higher order Padé's approximation is used, a higher order equation relating excess void reactivity with excess fuel temperature is obtained**, instead of the Eq. (3c) characteristic of March-Leuba's model.

So, let us generalize formula (5) for excess void reactivity, and from now on, let us work with a **generic impulse response function** $h_\alpha(t)$. With formula (5) in terms of a generalized impulse response function, the review of the generalization of March-Leuba's classical ROM for coupled thermo-hydraulic-neutronic instabilities is complete.

2.2 Derivation of a nonlinear integral-differential equation for the generalized model

For our purpose, it is better to introduce the logarithmic power: $x(t) = \ln\left(\frac{P(t)}{P_0}\right)$

Then $P = P_0 \cdot e^x$ and $P - P_0 = P_0 \cdot (e^x - 1)$. Besides: $\frac{\beta}{\Lambda} = \frac{\beta}{\Lambda} \cdot \int_0^\infty D(u) \cdot du$

Then Eq. (1) reduces to this one: $\frac{dx(t)}{dt} = \frac{\rho}{\Lambda} + \frac{\beta}{\Lambda} \cdot \int_0^\infty D(u) \cdot \left(e^{x(t-u)-x(t)} - 1\right) \cdot du$

If we define $g(x) = e^x - 1$, the last equation can be written this way:

$$\frac{dx(t)}{dt} = \frac{\rho}{\Lambda} + \frac{\beta}{\Lambda} \cdot \int_0^\infty D(u) \cdot g(x(t-u) - x(t)) \cdot du \qquad (8)$$

Substituting the excess reactivity from (3a) and (5) in (8) if follows:

$$\frac{dx(t)}{dt} = \frac{\left(\alpha_V \cdot \int_0^\infty h_\alpha(u) \cdot \Delta T(t-u) \cdot du + \alpha_T \cdot \Delta T\right)}{\Lambda} + \frac{\beta}{\Lambda} \cdot \int_0^\infty D(u) \cdot g(x(t-u) - x(t)) \cdot du \qquad (9)$$

Now, let us introduce the operator: $\hat{L}_T[\ ] = \frac{d}{dt}[\ ] + \frac{h}{C}[\ ]$ (10)

From the definition of the operator $\hat{L}_T$ and with $g(x) = e^x - 1$, Eq. (3b) can be recast as:

$$\hat{L}_T[\Delta T] = \frac{(P - P_0)}{C} = \frac{P_0}{C} \cdot g(x) \qquad (11)$$

Then, applying $\hat{L}_T$ to both members of Eq. (9) we obtain:

$$\hat{L}_T[\dot{x}] = \ddot{x} + \frac{h}{C}\cdot\dot{x} = \frac{1}{\Lambda}\cdot\hat{L}_T[\rho_\alpha] + \frac{1}{\Lambda}\cdot\alpha_T\cdot\frac{P_0}{C}\cdot g(x) + \frac{\beta}{\Lambda}\cdot\hat{L}_T\left[\int_0^\infty D(u)\cdot g(x(t-u)-x(t))\cdot du\right] \quad (12)$$

Now, let's keep in mind:

$$\hat{L}_T[\rho_\alpha] = \alpha_V\cdot\int_0^\infty h_\alpha(u)\cdot\hat{L}_T[\Delta T(t-u)]\cdot du = \frac{\alpha_V\cdot P_0}{C}\cdot\int_0^\infty h_\alpha(u)\cdot g(t-u)\cdot du \quad (13)$$

$$\hat{L}_T\left[\int_0^\infty D(u)\cdot g(x(t-u)-x(t))\cdot du\right] = \int_0^\infty D(u)\cdot\left[\frac{d}{dt}g[x(t-u)-x(t)] + \frac{h}{C}\cdot g[x(t-u)-x(t)]\right]\cdot du \quad (14)$$

Let us introduce:

A **thermal frequency**

$$\omega_T{}^2 = \frac{-\alpha_T\cdot P_0}{\Lambda\cdot C} = \frac{|\alpha_T|\cdot P_0}{\Lambda\cdot C} \quad (15a)$$

A **void frequency**

$$\omega_V{}^2 = \frac{-b_3\cdot P_0}{b_2\cdot C\cdot\Lambda} = \frac{|\alpha_V|\cdot P_0}{C\cdot\Lambda} \quad (15b)$$

Then, from (12) we derive the following **nonlinear integral-differential equation**:

$$\ddot{x} + \frac{h}{C}\cdot\dot{x} + \omega_T{}^2\cdot g(x) + \omega_V{}^2\cdot\int_0^\infty h_\alpha(u)\cdot g(x(t-u))\cdot du = \frac{\beta}{\Lambda}\cdot\int_0^\infty D(u)\cdot\left[\frac{d}{dt}g[x(t-u)-x(t)] + \frac{h}{C}\cdot g[x(t-u)-x(t)]\right]\cdot du \quad (16)$$

With equation (16) we have an integral-differential equation for power kinetics with feedback, in terms of logarithmic power, suitable for nonlinear stability analysis by KBM methods.

3 NONLINEAR STABILITY ANALYSIS APPLYING KBM METHOD

Linear stability analysis and digital simulations, done with the classical March-Leuba's model, shows that under proper circumstances the steady sate of the BWR can lose its stability in an oscillatory way. A stable limit cycle of oscillation of power and temperature appears in the reactor. In certain cases, even chaotic motions can be produced in the framework of the mathematical model, although they don't seem to be found in practice [9].

Let us study the possibility of sustained oscillations in the generalized model given by Eq. (16) applying the method of averaging due to Krylov, Bogoliubov and Mitropolski (**KBM method**). A thorough and in-depth introduction to nonlinear dynamics, including the asymptotic KBM method can be found in the book by Attlee-Jackson [11]. The classical book of Minorsky has a detailed and friendly exposition of this and other asymptotic methods [12]. A more recent reference considers some applications of KBM method to integral-differential equations and in general to delay-differential equations, that before were dispersed in highly specialized journals [13]. Besides, there are some approaches to the estimation of limit cycles amplitudes and frequencies that make use of energy methods [14].

To zero order, when the nonlinear response function $g(x)$ is not antisymmetric (so $g(x) \neq -g(-x)$) the KBM method begins with the ansatz (tentative solution) $x(t) \approx c(a(t)) + a(t) \cdot \cos\psi(t)$ to solve in a first approximation (the method allows obtain higher order approximations, but we don't need them in this paper) the following equation (the point over a function of time represents the time derivative of the function):

$$\ddot{x}(t) = \hat{F}[x, \dot{x}] \tag{17}$$

Here $\hat{F}[x, \dot{x}]$ is in general an integral-differential operator as the one that appears in Eq. (16) The phase is given by $\psi(t) = \omega_0 \cdot t + \theta(t)$, and the assumption is made that both $a(t)$ and $\theta(t)$ being **slowly varying functions of time** (time scales at least an order of magnitude greater than $\frac{2\pi}{\omega_0}$). During the construction of solutions of nonlinear integral-differential equations, the center frequency ω_0 is found, after determining the equation that gives the time variation of the phase, by the condition: $\omega_0 = \lim_{a \downarrow 0} \frac{d}{dt} \psi(t)$

3.1 Summary of the KBM perturbation method to zero order and two examples of its application to the construction of approximate solutions and to local bifurcation analysis in models with anti-symmetric response functions.

When the response function $g(x)$ is anti-symmetric (so $g(x) = -g(-x)$ for every x), the KBM method can be applied beginning with the simpler ansatz $x(t) \approx a(t) \cdot \cos\psi(t)$ to solve, in a first approximation, equation (17). This ansatz is the one that appears almost always in the literature.

As the application of KBM to differential equations with delays is less familiar to people in the nuclear engineering field, we begin with two simple examples of delayed nonlinear differential equations (**delayed van der Pol oscillators**).

The first one has a lumped delay in the response function:

$$\frac{d^2x(t)}{dt^2} + \alpha \cdot \left(x^2(t) - \mu\right) \cdot \frac{dx(t)}{dt} + \omega_0{}^2 \cdot x(t - \tau) = 0 \tag{18 a}$$

In (18a) the integral-differential operator $\hat{F}[x, \dot{x}]$ is given by:

$$\hat{F}[x, \dot{x}] = -\alpha \cdot \left(x^2(t) - \mu\right) \cdot \frac{dx(t)}{dt} - \omega_0{}^2 \cdot x(t - \tau)$$

The second example has a distributed delay in the response function:

$$\frac{d^2x(t)}{dt^2} + \alpha \cdot \left(x^2(t) - \mu\right) \cdot \frac{dx(t)}{dt} + \omega_0{}^2 \cdot \int_0^\infty h(u) \cdot x(t - u) \cdot du = 0 \tag{18b}$$

In (18 b) case the integral-differential operator $\hat{F}[x, \dot{x}]$ is given by:

$$\hat{F}[x,\dot{x}] = -\alpha \cdot \left(x^2(t) - \mu\right) \cdot \frac{dx(t)}{dt} - {\omega_0}^2 \cdot \int_0^\infty h(u) \cdot x(t-u) \cdot du$$

The response function in these cases is a linear one $g(x) = x$ It is anti-symmetric and appears in the integrand of a term that describes either a discrete delay effect in (18a) or a continuous distribution of delay effects in (18b).

Let us begin with the KBM **ansatz** $x(t) \approx a(t) \cdot \cos\psi(t)$ assuming that α is **positive** and small enough, and μ is **a real number**, not necessarily of small absolute value.

First main assumption of the KBM method: Let us suppose that $\psi(t) = \omega_* \cdot t + \theta(t)$, were **both the phase** $\theta(t)$ **and the amplitude** $a(t)$ **are slowly varying relative to** $\omega_* \cdot t$

To apply the KBM method in an oscillator case, the following restriction is imposed:

$$\frac{dx(t)}{dt} \approx \frac{da(t)}{dt} \cdot \cos\psi(t) - a(t) \cdot \sin\psi(t) \cdot \frac{d\psi}{dt} = -\omega_* \cdot a(t) \cdot \sin\psi(t) \qquad (19a)$$

As consequence of this last equation:

$$\frac{d^2x(t)}{dt^2} \approx -\omega_* \cdot \left(\frac{da(t)}{dt} \cdot \sin\psi(t) + a(t) \cdot \cos\psi(t) \cdot \frac{d\psi(t)}{dt} \right) \qquad (19b)$$

Substituting the ansatz in equations (18a) and (18b), and using equations (19) we obtain one system of two equations in the unknowns $\frac{da(t)}{dt}$ and $\frac{d\psi(t)}{dt}$ for each case (that is discrete delay in one case, continuous distribution of delays in the other case):

$$\frac{da(t)}{dt} \cdot \cos\psi(t) - a(t) \cdot \sin\psi(t) \cdot \frac{d\psi}{dt} = -\omega_* \cdot a(t) \cdot \sin\psi(t) \qquad (20a)$$

$$\frac{da(t)}{dt} \cdot \sin\psi(t) + a(t) \cdot \cos\psi(t) \cdot \frac{d\psi(t)}{dt} = -\frac{1}{\omega_*} \cdot \frac{d^2x(t)}{dt^2} =$$

$$= \begin{cases} \frac{\alpha}{\omega_*} \cdot \left(a^2(t) \cdot \cos^2\psi(t) - \mu\right) \cdot \left(-\omega_* \cdot a(t) \cdot \sin\psi(t)\right) + \frac{{\omega_0}^2}{\omega_*} \cdot \left(a(t-\tau) \cdot \cos\psi(t-\tau)\right) \\ \frac{\alpha}{\omega_*} \cdot \left(a^2(t) \cdot \cos^2\psi(t) - \mu\right) \cdot \left(-\omega_* \cdot a(t) \cdot \sin\psi(t)\right) + \frac{{\omega_0}^2}{\omega_*} \cdot \int_0^\infty h(u) \cdot a(t-u) \cdot \cos\psi(t-u) \cdot du \end{cases} \qquad (20b)$$

Second main assumption of the KBM method when applied to delay equations: Now we assume that $a(t-\tau) \approx a(t)$ and $\psi(t-\tau) \approx \psi(t) - \omega_* \cdot \tau$ in the discrete delay case, and for the continuous delay case $\int_0^\infty h(u) \cdot a(t-u) \cdot \cos\psi(t-u) \cdot du \approx a(t) \cdot \int_0^\infty h(u) \cdot \cos(\psi(t) - \omega_* \cdot u) \cdot du$

Taking all this into account, from equations (20) we derive the following equations for $\frac{da}{dt}$ and $\frac{d\psi}{dt}$:

$$\frac{da}{dt} \approx -\omega_* \cdot a \cdot \sin\psi \cdot \cos\psi - \frac{\alpha}{\omega_*} \cdot \left(a^2 \cdot \cos^2\psi - \mu\right) \cdot \left(-\omega_* \cdot a \cdot \sin^2\psi\right) + \\ + \frac{\omega_0^2}{\omega_*} \cdot a \cdot \left\{ \begin{array}{l} \left(\cos\psi \cdot \sin\psi \cdot \cos\omega_* \tau + \sin^2\psi \cdot \sin\omega_* \tau\right) \\ \left(\cos\psi \cdot \sin\psi \cdot \int_0^\infty h(u) \cos\omega_* u \cdot du + \sin^2\psi \cdot \int_0^\infty h(u) \sin\omega_* u \cdot du\right) \end{array} \right. \tag{21a}$$

$$a \cdot \frac{d\psi}{dt} \approx \omega_* \cdot a \cdot \sin^2\psi - \frac{\alpha}{\omega_*} \cdot \left(a^2 \cdot \cos^2\psi(t) - \mu\right) \cdot \left(-\omega_* \cdot a \cdot \sin\psi \cdot \cos\psi\right) + \\ + \frac{\omega_0^2}{\omega_*} \cdot a \cdot \left\{ \begin{array}{l} \left(\cos^2\psi \cdot \cos\omega_* \tau + \sin\psi \cdot \cos\psi \cdot \sin\omega_* \tau\right) \\ \left(\cos^2\psi \cdot \int_0^\infty h(u) \cos\omega_* u \cdot du + \sin\psi \cdot \cos\psi \cdot \int_0^\infty h(u) \sin\omega_* u \cdot du\right) \end{array} \right. \tag{21b}$$

Third main assumption of the KBM method: In order to obtain the differential equations for the dynamics of the amplitude and phase of the oscillations we take averages relative to ψ between 0 and $2 \cdot \pi$

Taking the averages, it follows:

$$\frac{da}{dt} \approx \frac{\alpha}{2} \cdot a \cdot \left(\mu - \frac{a^2}{4}\right) + \frac{\omega_0^2}{2 \cdot \omega_*} \cdot a \cdot \left\{ \begin{array}{l} \sin\omega_* \tau \\ \int_0^\infty h(u) \sin\omega_* u \cdot du \end{array} \right. =$$

$$= \left\{ \begin{array}{l} \frac{\alpha}{8} \cdot a \cdot \left(4 \cdot \left(\mu + \frac{\omega_0^2}{\alpha \cdot \omega_*} \cdot \sin\omega_* \tau\right) - a^2\right) \\ \frac{\alpha}{8} \cdot a \cdot \left(4 \cdot \left(\mu + \frac{\omega_0^2}{\alpha \cdot \omega_*} \cdot \int_0^\infty h(u) \sin\omega_* u \cdot du\right) - a^2\right) \end{array} \right. \tag{22a}$$

$$\frac{d\psi}{dt} \approx \frac{\omega_*}{2} + \frac{\omega_0^2}{2 \cdot \omega_*} \cdot \left\{ \begin{array}{l} \cos\omega_* \tau \\ \int_0^\infty h(u) \cos\omega_* u \cdot du \end{array} \right. \tag{22b}$$

Thus, we obtain:

Case 1: discrete time delay

$$\frac{da}{dt} \approx \frac{\alpha}{8} \cdot a \cdot \left(4 \cdot \left(\mu + \frac{\omega_0^2}{\alpha \cdot \omega_*} \cdot \sin \omega_* \tau \right) - a^2 \right) \tag{23a}$$

$$\frac{d\psi}{dt} \approx \frac{\omega_*}{2} + \frac{\omega_0^2}{2 \cdot \omega_*} \cdot \cos \omega_* \tau \tag{23b}$$

Case 2: continuous distribution of time delays

$$\frac{da}{dt} \approx \frac{\alpha}{8} \cdot a \cdot \left(4 \cdot \left(\mu + \frac{\omega_0^2}{\alpha \cdot \omega_*} \cdot \int_0^\infty h(u) \sin \omega_* u \cdot du \right) - a^2 \right) \tag{24a}$$

$$\frac{d\psi}{dt} \approx \frac{\omega_*}{2} + \frac{\omega_0^2}{2 \cdot \omega_*} \cdot \int_0^\infty h(u) \cos \omega_* u \cdot du \tag{24b}$$

Now we return to the main assumption of KBM method, $\psi(t) = \omega_* \cdot t + \theta(t)$, being $a(t)$ and $\theta(t)$ slowly varying functions in comparison with $\omega_* \cdot t$

From (23b) and (24b) it follows that $\frac{d\psi}{dt}$ is constant. So $\frac{d\psi}{dt}$ must be equal to ω_*.

The meaning of this result is that **in a first approximation** the phase varies with constant angular velocity ω_*

Then for the discrete delay we have:

$$\omega_*^2 \approx \omega_0^2 \cdot \cos(\omega_* \cdot \tau) \tag{25a}$$

And for the continuous distribution of delays:

$$\omega_*^2 \approx \omega_0^2 \cdot \int_0^\infty h(u) \cos \omega_* u \cdot du \tag{25b}$$

Equation (25a) always has a solution $0 < \omega_* < \frac{\pi}{2 \cdot \tau}$ for all possible delays τ.

For the continuous distribution of time delays, the distribution function $h(u)$ must be non-negative and must verify $\int_0^\infty h(u) \cdot du = 1$

As consequence for any ω we always have $\int_0^\infty h(u) \cdot \cos(\omega \cdot u) \cdot du < 1$ so equation (25b) always has a unique solution. For example, if $h(u) = \lambda \cdot e^{-\lambda \cdot u}$, then $\int_0^\infty h(u) \cos \omega_* u \cdot du = \frac{\lambda^2}{\lambda^2 + \omega_*^2}$ and $\int_0^\infty h(u) \sin \omega_* u \cdot du = \frac{\lambda \cdot \omega_*}{\lambda^2 + \omega_*^2}$

From (25b) we derive $$\omega_* = \sqrt{\frac{\sqrt{\lambda^4 + 4\cdot\lambda^2\cdot\omega_0} - \lambda^2}{2}} \qquad (26)$$

In any case, for suitably restricted delays we solve equations (25a) or (25b) in order to find **the fast frequency** ω_*.

If $\omega_*\cdot\tau$ is small enough, we can approximate $\cos(\omega_*\cdot\tau) \approx 1 - \frac{(\omega_*\cdot\tau)^2}{2}$

From (25a) we obtain: $$\omega_* \approx \frac{\omega_0}{\sqrt{1+(\omega_0\cdot\tau)^2}} \qquad (27)$$

If there is no delay, $\tau = 0$ and $\omega_* = \omega_0$ as it should be.

Now, **remember that we take α positive**.

When: $\begin{cases} \mu + \frac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\sin\omega_*\tau < 0 \\ \mu + \frac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\int_0^\infty h(u)\cdot\sin\omega_* u\cdot du < 0 \end{cases}$ the rest point of the oscillator is stable.

In fact, from equations (23a) and (24a) we deduce that it is **globally and asymptotically stable**.

However, when: $\begin{cases} \mu + \frac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\sin\omega_*\tau > 0 \\ \mu + \frac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\int_0^\infty h(u)\cdot\sin\omega_* u\cdot du > 0 \end{cases}$ the rest point of the oscillator is **unstable** and **a stable limit cycle** appears (**supercritical Hopf bifurcation**).

If $a_0 = a(0)$ is the initial value of the amplitude of oscillation, according to equations (23a) and (24a) the amplitude (slowly varying relative to the time scale $T_* = \frac{2\pi}{\omega_*}$) is given by:

$$a(t) \approx \frac{a_*}{\sqrt{1-\left(1-\frac{a_*^{\,2}}{a_0^{\,2}}\right)\cdot\exp\left[-\frac{\alpha\cdot a_*^{\,2}}{4}\cdot t\right]}} \qquad (28)$$

The **radius of the stable limit cycle** is given by the approximate expression:

$$a_* \approx \begin{cases} 2\cdot\sqrt{\mu+\dfrac{\omega_0}{\alpha}\cdot\sin(\omega_*\cdot\tau)} \\ 2\cdot\sqrt{\mu+\dfrac{\omega_0}{\alpha}\cdot\int_0^\infty h(u)\cdot\sin(\omega_*\cdot u)\cdot du} \end{cases} \tag{29}$$

When the discrete delay τ is between 0 y $\dfrac{\pi}{2\cdot\omega_0}$ the limit cycle radius increases with the delay from $2\cdot\sqrt{\mu}$ up to $2\cdot\sqrt{\mu+\dfrac{\omega_0}{\alpha}}$

So, we have here **a bifurcation parameter** which is $\mu+\dfrac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\sin\omega_*\tau$ in the discrete delay case, and $\mu+\dfrac{\omega_0^2}{\alpha\cdot\omega_*}\cdot\int_0^\infty h(u)\cdot\sin\omega_* u\cdot du$ in the continuous delay case. We see that this bifurcation parameter is a function of the delay. If $\tau=0$, we have the well-known van der Pol equation, and now the bifurcation parameter is simply μ.

If $\mu=1$, (29) it follows $a_*=2$. This is an already well-known result.

3.2 Application of KBM method to the integral-differential equation of the BWR model

Now, beginning with $x(t)\approx c(a(t))+a(t)\cdot\cos\psi(t)$, the following restriction is imposed:

$$\dot{x}(t)\approx\left(\frac{dc(a(t))}{da}+\cos\psi(t)\right)\cdot\dot{a}(t)-a(t)\cdot sen\,\psi(t)\cdot\dot{\psi}(t)=-\omega_0\cdot a(t)\cdot sen\,\psi(t) \tag{28}$$

Then:
$$\ddot{x}(t)\approx-\omega_0\cdot\left(\dot{a}(t)\cdot sen\,\psi(t)+a(t)\cdot\cos\psi(t)\cdot\dot{\psi}(t)\right)=\hat{F}[x,\dot{x}] \tag{29}$$

In our case

$$\begin{aligned}\hat{F}[x,\dot{x}]=&-\frac{h}{C}\cdot\dot{x}-{\omega_T}^2\cdot g(x)-{\omega_V}^2\cdot\int_0^\infty h_\alpha(u)\cdot g(x(t-u))\cdot du+\\ &\frac{\beta}{\Lambda}\cdot\int_0^\infty D(u)\cdot\left[\frac{d}{dt}g[x(t-u)-x(t)]+\frac{h}{C}\cdot g[x(t-u)-x(t)]\right]\end{aligned} \tag{30}$$

The first step is the construction of an approximate relation $c(a)$ between the offset c and the amplitude of oscillation a.

This can be done substituting the ansatz $x(t)\approx c(a)+a\cdot\cos\psi$ in $g(x)$ and writing the following equation (reference [14]):
$$\int_0^{2\pi} g(c+a\cdot\cos\psi)\cdot d\psi=0 \tag{31}$$

Now, it is known that $g(a\cdot\cos\psi)=e^{a\cdot\cos\psi}-1=(I_0(a)-1)+2\cdot\sum_{n=1}^{\infty}I_n(a)\cdot\cos(n\psi)$ (see reference [15]) where the functions $I_n(a)$ are Bessel functions of the second kind.

So $$g(c+a\cdot\cos\psi)=e^{c+a\cdot\cos\psi}-1=\left(e^{c}\cdot I_0(a)-1\right)+2\cdot e^{c}\cdot\sum_{n=1}^{\infty}I_n(a)\cdot\cos(n\psi) \qquad (32)$$

From (31) and (32) we derive: $\left(e^{c}\cdot I_0(a)-1\right)=0$ so $$c=\log_e\left(\frac{1}{I_0(a)}\right) \qquad (33)$$

Then, having determined the offset $c(a)$, after substituting $x(t)\approx c(a(t))+a(t)\cdot\cos\psi(t)$ and $\dot{x}(t)\approx-\omega_0\cdot a(t)\cdot\sin\psi(t)$ in formula (30) for $\hat{F}[x,\dot{x}]$, and inserting the result in (18), we can solve the resulting equations (28) and (29) to obtain:

$$\left(1+\frac{dc}{da}\cdot\cos\psi\right)\cdot\dot{a}=-\omega_0\cdot a\cdot\sin\psi\cdot\cos\psi+\frac{1}{\omega_0}\cdot\hat{F}[a\cdot\cos\psi,-\omega_0\cdot a\cdot\sin\psi]\cdot\sin\psi \qquad (34a)$$

$$\dot{\psi}-\left(\cos\psi\cdot\sin\psi+\frac{dc}{da}\cdot\cos\psi\right)\cdot\left(\frac{\dot{a}}{a}\right)=\omega_0\cdot\sin^2\psi-\frac{1}{a\cdot\omega_0}\cdot\hat{F}[a\cdot\cos\psi,-\omega_0\cdot a\cdot\sin\psi]\cdot\cos\psi \qquad (34b)$$

As we suppose that the functions $c(a(t))$, $a(t)$ and $\psi(t)-\omega_0\cdot t$ vary slowly during a period $T_0=\frac{2\pi}{\omega_0}$, a moving time average $\frac{1}{T_0}\cdot\int_{t}^{t+T_0}(\,)\cdot dt'$ can be applied to both members of (34a) and (34b).

The quotients $\frac{a(t+T_0)-a(t)}{T_0}$ and $\frac{\psi(t+T_0)-\psi(t)}{T_0}$ are approximated by $\frac{d}{dt}a(t)=\dot{a}(t)$ and $\frac{d}{dt}\psi(t)=\dot{\psi}(t)$ respectively. The next step is to realize that $d\psi\approx\omega_0\cdot dt$, so it is possible to make the approximation $\frac{1}{T_0}\cdot\int_{t}^{t+T_0}(\,)\cdot dt'\approx\frac{1}{2\pi}\cdot\int_{\psi}^{\psi+2\pi}(\,)\cdot d\psi'$ Thus, we obtain, after considering the periodicities of the sine and cosine functions, the following equations of evolution:

$$\dot{a}=-\frac{1}{2\pi\cdot\omega_0}\cdot\int_0^{2\pi}\hat{F}[c+a\cdot\cos\psi,-\omega_0\cdot a\cdot\sin\psi]\cdot\sin\psi\cdot d\psi \qquad (35a)$$

$$\dot{\psi}=\frac{1}{2}\omega_0-\frac{1}{2\pi\cdot\omega_0\cdot a}\cdot\int_0^{2\pi}\hat{F}[c+a\cdot\cos\psi,-\omega_0\cdot a\cdot\sin\psi]\cdot\cos\psi\cdot d\psi \qquad (35b)$$

Let us calculate the integrals in (35a) and (35b) for the reactor model given by Eq. (16).

Given the ansatz $x(t)\approx c(a(t))+a(t)\cdot\cos\psi(t)$ and taking into account the slow variation of the offset $c(a(t))$, the amplitude $a(t)$ and phase difference $\theta(t)=\psi-\omega_0\cdot t$ during a period $\frac{2\pi}{\omega_0}$, it is possible to put

$$x(t-u)\approx c(a(t))+a(t)\cdot\cos(\psi(t)-\omega_0\cdot u) \qquad (36a)$$

$$x(t-u)-x(t)\approx-a(t)\cdot U(\omega_0\cdot u)\cdot\cos\left(\psi(t)+\phi(\omega_0\cdot u)\right) \qquad (36b)$$

Here, by definition:

$$U\cdot\cos\phi=1-\cos(\omega_0\cdot u) \qquad (36c)$$

$$U\cdot\sin\phi=\sin(\omega_0\cdot u) \qquad (36d)$$

In addition, we need the following results that can be found in reference [15]:

$$I_n(a)=\left(\frac{a}{2}\right)^n\cdot\left(\frac{1}{n!}+\frac{a^2}{4\cdot(n+1)!}+\frac{a^4}{32\cdot(n+2)!}+....\right)\qquad n=0,1,2,3,\cdots \tag{37a}$$

$$I_0(a)-I_2(a)=\frac{2\cdot I_1(a)}{a}=1+\frac{a^2}{8}+\frac{a^4}{192}+... \tag{37b}$$

$$I_0(a)=1+\frac{a^2}{4}+\frac{a^4}{64}+\cdots \tag{37c}$$

$$I_1(a)=\frac{a}{2}\cdot\left(1+\frac{a^2}{8}+\frac{a^4}{192}+\cdots\right)=\frac{a}{2}+\frac{a^3}{16}+\frac{a^5}{384}+.... \tag{37d}$$

$$\frac{2\cdot I_1(a)}{a\cdot I_0(a)}=1-\frac{I_2(a)}{I_0(a)}=\frac{\left(1+\frac{a^2}{8}+\frac{a^4}{192}+\cdots\right)}{\left(1+\frac{a^2}{4}+\frac{a^4}{64}+\cdots\right)}=1-\frac{a^2}{8}+\frac{a^4}{48}\pm\cdots \tag{37e}$$

$$I_2(a)=\left(\frac{a}{2}\right)^2\cdot\left(\frac{1}{2}+\frac{a^2}{24}+\frac{a^4}{768}+....\right)=\frac{a^2}{8}+\frac{a^4}{96}+.... \tag{37f}$$

From (32) and (33):

$$g\left(c+a\cdot\cos(\psi-\omega_0\cdot u)\right)=2\cdot\left(\frac{1}{I_0(a)}\right)\cdot\sum_{n=1}^{\infty}I_n(a)\cdot\cos n\cdot(\psi-\omega_0\cdot u) \tag{38a}$$

From $g(a\cdot\cos\psi)=e^{a\cdot\cos\psi}-1=(I_0(a)-1)+2\cdot\sum_{n=1}^{\infty}I_n(a)\cdot\cos(n\psi)$ substituting a by $-a\cdot U$ and ψ by $\psi+\phi$, taking into account that, from formula (37a), it can be deduced that $I_n(-a)=(-1)^n\cdot I_n(a)$ it follows:

$$g\left(-a\cdot U\cdot\cos(\psi+\phi)\right)=\left(I_0(a\cdot U)-1\right)+2\cdot\sum_{n=1}^{\infty}(-1)^n I_n(a\cdot U)\cdot\cos n\cdot(\psi+\phi) \tag{38b}$$

Furthermore, because the time variation of the phase ψ is by hypothesis fast relative to the time variation of the amplitude a we have:

$$\frac{d}{dt}a\cdot U\cdot\cos(\psi+\phi)\cong-\omega_0\cdot a\cdot U\cdot\sin(\psi+\phi) \tag{38c}$$

As consequence of (38c) we obtain:

$$\frac{d}{dt}g\left(-a\cdot U\cdot\cos(\psi+\phi)\right)\cong-\omega_0\cdot a\cdot U\cdot\sin(\psi+\phi)\cdot e^{-a\cdot U\cdot\cos(\psi+\phi)} \tag{38d}$$

From equations (36), (37) and (38) the following nonlinear **equations of evolution** may be derived by a straightforward but lengthy calculation of the integrals in (35a) and (35b):

$$\frac{da}{dt}=\frac{a}{2}\cdot\left[-\frac{h}{C}+\frac{\omega_V^2}{\omega_0}\cdot H_{\alpha,s}(\omega_0)\cdot\left(\frac{2I_1(a)}{a\cdot I_0(a)}\right)+\frac{\beta}{\Lambda}\cdot\left\{\begin{array}{l}\frac{h}{\omega_0\cdot C}\cdot\int_0^{+\infty}D(u)\cdot\left(\frac{2\cdot I_1(aU)}{aU}\right)\cdot U\sin\phi\cdot du\\ -\int_0^{+\infty}D(u)\cdot I_0(aU)\cdot U\cos\phi\cdot du\end{array}\right\}\right] \tag{39a}$$

$$\frac{d\psi}{dt}=\frac{\omega_0}{2}+\left\{\begin{array}{l}\dfrac{\left(\omega_T^2+\omega_V^2\cdot H_{\alpha,c}(\omega_0)\right)}{2\cdot\omega_0}\cdot\left(\dfrac{2\cdot I_1(a)}{a\cdot I_0(a)}\right)+\\ \dfrac{\beta}{\Lambda}\cdot\left[\displaystyle\int_0^{+\infty}D(u)\cdot I_0(aU)\cdot U\sin\phi\cdot du-\frac{h}{\omega_0\cdot C}\cdot\int_0^{+\infty}D(u)\cdot\left(\frac{2\cdot I_1(aU)}{aU}\right)\cdot U\cos\phi\cdot du\right]\end{array}\right\} \tag{39b}$$

By definition:
$$H_{\alpha,.c}(\omega_0)=\int_0^{\infty}h_\alpha(u)\cdot\cos(\omega_0 u)\cdot du \tag{40a}$$

$$H_{\alpha,.s}(\omega_0)=\int_0^{\infty}h_\alpha(u)\cdot\sin(\omega_0 u)\cdot du \tag{40b}$$

Now, we introduce:
$$D_c(\omega_0)=\int_0^{\infty}D(u)\cdot\cos(\omega_0 u)\cdot du \tag{40c}$$

$$D_s(\omega_0)=\int_0^{\infty}D(u)\cdot\sin(\omega_0 u)\cdot du \tag{40d}$$

From the development of Bessel functions of the second kind in series of powers of the amplitude of oscillation (equations (37)) and from formulae (39a) and (39b) it follows that:

$$\frac{da}{dt}=a\cdot\left(\kappa_0+\kappa_2\cdot a^2+\kappa_4\cdot a^4+\cdots\right) \tag{41a}$$

$$\frac{d\psi}{dt}=\omega_0+\delta_2\cdot a^2+\delta_4\cdot a^4+\cdots \tag{41b}$$

The fast frequency ω_0 in the limit of small amplitudes can be determined from the equation (39b) by the following limit, where $\omega(a)=\dfrac{d\psi}{dt}$ is the instantaneous frequency of power oscillations, given as a function of the amplitude: $\omega_0=\lim_{a\downarrow 0}\omega(a)$ (42)

From (39b) and (42), jointly with the properties of Bessel functions given in equations (37), the definitions (40c) and (40d), and the relations (36c), (36d) between U, ϕ and $\omega_0 u$ it follows:
$$\omega_0{}^2=\omega_T{}^2+\omega_V{}^2\cdot H_{\alpha,.c}(\omega_0)+\frac{\beta}{\Lambda}\cdot\left(\omega_0\cdot D_s(\omega_0)-\frac{2\cdot h}{C}\cdot\left(1-D_c(\omega_0)\right)\right) \tag{43}$$

This equation, implicit in ω_0, allows us to find this frequency.

From (39a) the coefficient κ_0 that determines the stability of this steady state under small perturbations is obtained, being $\dot a=\dfrac{da}{dt}$:

$$2\cdot\kappa_0=2\cdot\lim_{a\downarrow 0}\left(\frac{\dot a}{a}\right)=-\frac{h}{C}+\frac{\omega_V{}^2}{\omega_0}H_{\alpha,.s}(\omega_0)+\frac{\beta}{\Lambda}\cdot\left(\left(\frac{h}{\omega_0\cdot C}\right)\cdot D_s(\omega_0)-\left(1-D_c(\omega_0)\right)\right) \tag{44}$$

The coefficients $\kappa_0,\kappa_2,\kappa_4$ in the development of $\dfrac{da}{dt}$ in series of powers of a can be written this way:

$$2\cdot\kappa_0 = -\frac{h}{C}+\frac{{\omega_V}^2}{\omega_0}H_{\alpha,,s}(\omega_0)+\frac{\beta}{\Lambda}\cdot\left(\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(0)-\mathrm{I}_c(0)\right) \tag{45a}$$

$$8\cdot\kappa_2 = \left\{-\frac{{\omega_V}^2}{\omega_0}H_{\alpha,,s}(\omega_0)+\frac{\beta}{\Lambda}\cdot\left(\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(1)-2\cdot \mathrm{I}_c(1)\right)\right\} \tag{45b}$$

$$48\cdot\kappa_4 = \left\{\frac{{\omega_V}^2}{\omega_0}H_{\alpha,,s}(\omega_0)+\frac{\beta}{\Lambda}\cdot\left(\frac{1}{4}\cdot\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(2)-\frac{3}{4}\mathrm{I}_c(2)\right)\right\} \tag{45c}$$

The integrals $I_s(n)$ and $I_c(n)$ related with the production of delayed neutrons that appear in (45a), (45b), (45c) are given by the following formulae, for $n=0$, $n=1$, $n=2$:

$$\mathrm{I}_c(n) = \int_0^{+\infty} D(u)\cdot U^{2n}\cdot U\cos\varphi\cdot du = 2^n\int_0^{+\infty} D(u)\cdot\left(1-\cos(\omega_0\cdot u)\right)^{2n+1}\cdot du \tag{46a}$$

$$\mathrm{I}_s(n) = \int_0^{+\infty} D(u)\cdot U^{2n}\cdot U\sin\varphi\cdot du = 2^n\int_0^{+\infty} D(u)\cdot\left(1-\cos(\omega_0\cdot u)\right)^{n}\cdot\sin(\omega_0\cdot u)\cdot du \tag{46b}$$

From (36c) and (36d) we see that $U\cdot\cos\phi = 1-\cos(\omega_0\cdot u)$ is always non-negative and oscillates between 0 and 2, while $U\cdot\sin\phi = \sin(\omega_0\cdot u)$ oscillates between -1 and 1.

So, $U^2 = 2\cdot(1-\cos(\omega_0\cdot u))$ is always positive and oscillates between 0 and 4.

Analogous formulae can be derived for κ_6 and the higher order parameters that appear in (41a), as well as for the parameters δ_2 and the higher order parameters that appear in (41b).

The parameters δ_2, δ_4 are given by the formulae:

$$16\cdot\delta_2 = -\frac{\left(\omega_T^2+\omega_V^2\cdot H_{\alpha,c}(\omega_0)\right)}{\omega_0}+\frac{4\cdot\beta}{\Lambda}\cdot\left[I_s(1)-\frac{h}{2\cdot\omega_0\cdot C}\cdot I_c(1)\right] \tag{47a}$$

$$96\cdot\delta_4 = -\frac{\left(\omega_T^2+\omega_V^2\cdot H_{\alpha,c}(\omega_0)\right)}{\omega_0}+\frac{\beta}{\Lambda}\cdot\left[\frac{2}{3}\cdot I_s(2)-\frac{h}{3\cdot\omega_0\cdot C}\cdot I_c(2)\right] \tag{47b}$$

In equations (46): $1-\cos(\omega_0\cdot u) = 2\cdot\sin^2\left(\frac{\omega_0\cdot u}{2}\right)$ $\sin(\omega_0\cdot u) = 2\cdot\sin\left(\frac{\omega_0\cdot u}{2}\right)\cdot\cos\left(\frac{\omega_0\cdot u}{2}\right)$

For one group of delayed neutron emitters $D(t) = \lambda\cdot e^{-\lambda\cdot t}$ and in this case:

$$\mathrm{I}_c(n) = 16\lambda\cdot\int_0^{+\infty} e^{-\lambda\cdot u}\cdot\left(\sin\left(\frac{\omega_0\cdot u}{2}\right)\right)^{4n+2}.du \tag{48a}$$

$$\mathrm{I}_s(n) = 16\lambda\cdot\int_0^{+\infty} e^{-\lambda\cdot u}\cdot\left(\sin\left(\frac{\omega_0\cdot u}{2}\right)\right)^{4n+1}.\cos\left(\frac{\omega_0\cdot u}{2}\right)\cdot du \tag{48b}$$

From (46a) and (46b) these formulae are obtained for a single group of delayed neutrons:

$$\mathrm{I}_c(0) = 1-D_c(\omega_0) = 1-\frac{\lambda^2}{\lambda^2+{\omega_0}^2} = \frac{{\omega_0}^2}{\lambda^2+{\omega_0}^2} \tag{49a}$$

$$\mathrm{I}_s(0) = D_s(\omega_0) = \frac{\lambda\cdot\omega_0}{\lambda^2+{\omega_0}^2} \tag{49b}$$

Substituting (49a) and (49b) in (43) we have the following formulae for the frequency ω_0 of small amplitude of oscillation around the steady state (corresponding to the power P_0):

$$\omega_0{}^2 = \omega_T{}^2 + \omega_V{}^2 \cdot H_{\alpha,.c}(\omega_0) - \frac{\left(\frac{\beta}{\Lambda}\right)\cdot\left(\frac{h}{C}-\lambda\right)\cdot\omega_0{}^2}{\lambda^2+\omega_0{}^2} \qquad (50a)$$

Taking representative values of $\frac{h}{C}(0.23\ \text{s}^{-1}), \lambda\ (0.08\ \text{s}^{-1}),\ \beta = 0,0056$ and $\Lambda = 4\times10^{-5}\,s$ it results: $\frac{\beta}{\Lambda}\cdot\left(\frac{h}{C}-\lambda\right) = 21 s^{-2}$. [10] [17]

Equation (50a) can be rewritten to facilitate the analysis:

$$\omega_0{}^2 = \frac{\omega_T{}^2 + \omega_V{}^2 \cdot H_{\alpha,.c}(\omega_0)}{1 + \frac{\left(\frac{\beta}{\Lambda}\right)\cdot\left(\frac{h}{C}-\lambda\right)}{\lambda^2+\omega_0{}^2}} \qquad (50b)$$

Given the estimated value of $\frac{\beta}{\Lambda}\cdot\left(\frac{h}{C}-\lambda\right)$ and the feasible values of ω_0 [1] [4], this equation can be approximated by the following one:

$$\left(\frac{\omega_0}{\lambda}\right)^2 \cong \frac{\omega_T{}^2 + \omega_V{}^2 \cdot H_{\alpha,.c}(\omega_0)}{\left(\frac{\beta}{\Lambda}\right)\cdot\left(\frac{h}{C}-\lambda\right)} \qquad (50c)$$

For only one group of delayed neutron emitters, (44a) reduces to:

$$2\cdot\kappa_0 = -\frac{h}{C} + \frac{\omega_V{}^2}{\omega_0} H_{\alpha,.s}(\omega_0) + \frac{\beta}{\Lambda}\cdot\left(\frac{\frac{h}{C}\cdot\lambda - \omega_0{}^2}{\lambda^2+\omega_0{}^2}\right) \qquad (51)$$

Now, let us focus on $H_{\alpha,.c}(\omega_0)$ and $H_{\alpha,.s}(\omega_0)$, given by equations (40a) and (40b) in terms of the generalized impulse response function $h_\alpha(t)$. We see that $H_{\alpha,.c}(\omega_0)$ is the cosine and $H_{\alpha,.s}(\omega_0)$ is the sine Fourier transforms of the generic function $h_\alpha(t)$.

For the special case of March-Leuba's model with three parameters b_1, b_2, b_3 in the transference function $\overline{H}_\alpha(s)$ from temperature to void reactivity, we obtain this cosine and sine Fourier transforms directly from equation (4), factorizing the static void coefficient $\alpha_V = \frac{b_3}{b_2}$ and putting $s = -j\cdot\omega_0$:

$$\overline{H}_\alpha(j\omega_0) = \alpha_V \cdot \frac{b_2}{b_2 - \omega_0{}^2 - jb_1\cdot\omega_0} \qquad (52a)$$

Then the real and imaginary parts give respectively the cosine and sine transforms:

$$H_{\alpha,c}(\omega_0)=\frac{b_2\cdot\left(b_2-\omega_0{}^2\right)}{\Delta(\omega_0)} \tag{52b}$$

$$H_{\alpha,s}(\omega_0)=\frac{\omega_0\cdot b_2\cdot b_1}{\Delta(\omega_0)} \tag{52c}$$

$$\Delta(\omega_0)=\left(b_2-\omega_0{}^2\right)^2+b_1{}^2\cdot\omega_0{}^2 \tag{52d}$$

By equations (7) the parameters b_1, b_2 and b_3 can be expressed in terms of the residence time of steam bubbles in the channel $\tau\approx\frac{H}{\overline{V_0}}$ and the channel height H. But $\overline{V_0}$ is a mean void propagation velocity proportional to coolant flow F in the fuel channels, so $\tau\propto\frac{1}{F}$.

All this, jointly with the formulae for the thermal frequency (15a) and void frequency (15b) ($\omega_T{}^2=\frac{-\alpha_T\cdot P_0}{\Lambda\cdot C}=\frac{|\alpha_T|\cdot P_0}{\Lambda\cdot C}$ and $\omega_V{}^2=\frac{-b_3\cdot P_0}{b_2\cdot C\cdot\Lambda}=\frac{|\alpha_V|\cdot P_0}{C\cdot\Lambda}$ respectively) allows us to give the parameters that appear in equations (39) for the amplitude and the phase velocities of the oscillations, as well defined analytic functions of the steady state power, coolant flow and other relevant parameters from reactor's thermal-hydraulics and neutronics.

From equations (15) and (50c) the following formula is obtained for the frequency of small amplitude oscillations:

$$\frac{\omega_0}{\lambda}=\sqrt{\frac{\left(|\alpha_T|+|\alpha_V|\cdot H_{\alpha,c}(\omega_0)\right)\cdot\left(\frac{P_0}{\Lambda\cdot C}\right)}{\frac{\beta}{\Lambda}\cdot\left(\frac{h}{C}-\lambda\right)}} \tag{53}$$

For the three parameters model, from (53) and (52b) we derive the following implicit equation for the frequency in the small amplitude limit:

$$\frac{\omega_0}{\lambda}=\sqrt{\frac{\left(|\alpha_T|+|\alpha_V|\cdot\frac{b_2\cdot\left(b_2-\omega_0{}^2\right)}{\Delta(\omega_0)}\right)\cdot\left(\frac{P_0}{\Lambda\cdot C}\right)}{\frac{\beta}{\Lambda}\cdot\left(\frac{h}{C}-\lambda\right)}} \tag{54}$$

For the three parameters model of the transference function relating temperature to void reactivity, from equations (45a), (45b), (45c) and (52c) we obtain the following formulae:

$$2\cdot\kappa_0=-\frac{h}{C}+\left(\frac{b_1\cdot b_2\cdot|\alpha_V|}{\Delta(\omega_0)\cdot C\cdot\Lambda}\right)\cdot P_0+\frac{\beta}{\Lambda}\cdot\left(\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(0)-\mathrm{I}_c(0)\right) \tag{55a}$$

$$8\cdot\kappa_2=\left\{-\left(\frac{b_1\cdot b_2\cdot|\alpha_V|}{\Delta(\omega_0)\cdot C\cdot\Lambda}\right)\cdot P_0+\frac{\beta}{\Lambda}\cdot\left(\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(1)-2\cdot \mathrm{I}_c(1)\right)\right\} \tag{55b}$$

$$48\cdot\kappa_4=\left\{\left(\frac{b_1\cdot b_2\cdot|\alpha_V|}{\Delta(\omega_0)\cdot C\cdot\Lambda}\right)\cdot P_0+\frac{\beta}{\Lambda}\cdot\left(\frac{1}{4}\cdot\left(\frac{h}{\omega_0\cdot C}\right)\cdot \mathrm{I}_s(2)-\frac{3}{4}\mathrm{I}_c(2)\right)\right\} \tag{55c}$$

In these formulae $\mathrm{I}_c(n)$ and $\mathrm{I}_s(n)$ are given by formulae (46a) and (46b) with $n=0,1,2$.

If there is only group of delayed neutrons in the mathematical model, formula for the three parameters model (55a) reduces to:

$$2\cdot\kappa_0=-\frac{h}{C}+\left(\frac{b_1\cdot b_2\cdot|\alpha_V|}{\Delta(\omega_0)\cdot C\cdot\Lambda}\right)\cdot P_0+\frac{\beta}{\Lambda}\cdot\left(\frac{\frac{h}{C}\cdot\lambda-{\omega_0}^2}{\lambda^2+{\omega_0}^2}\right) \tag{56}$$

3.3 Power and logarithmic power.

From the definition of the dimensionless logarithmic power $x(t)=\log_e\frac{P(t)}{P_0}$, from the ansatz for the logarithmic power $x(t)=c(a(t))+a(t)\cdot\cos\psi(t)$ and from formula (33) for the offset, the following formula is obtained for the nuclear reactor power $P(t)$:

$$\frac{P(t)}{P_0}=\frac{1}{I_0(a(t))}\cdot\exp\left[a(t)\cdot\cos\psi(t)\right] \tag{57a}$$

The Bessel function of second kind and zero order is given by formula $I_0(a)=1+\frac{a^2}{4}+\frac{a^4}{64}+\cdots$

When the dimensionless amplitude of oscillation is small enough relative to 1, the offset may be neglected. Then the ansatz for the logarithmic power reduces to the simpler tentative solution used in section 3.1 of this paper, and in reference [7].

If the logarithmic power follows a harmonic motion with constant amplitude and frequency the nuclear reactor power varies as shown in the next formula:

$$P(t)=\frac{P_0}{I_0(a_s)}\cdot\exp\left[a_s\cdot\cos\left(\omega(a_s)\cdot t\right)\right] \tag{57b}$$

A stable power oscillation like this one could be associated to a stable limit cycle of oscillation in the logarithmic power of the reactor. This kind of cycle appears in the framework of a supercritical Poincaré-Andronov-Hopf (PAH) bifurcation or in a Bautin bifurcation scenario, when the steady state crosses certain subsets of a stability boundary in parameters space. This will be considered in the next section.

3.4 Stability boundary. Draft of the possibility of soft and hard excitation of direct-loop power oscillations

Now, let us qualitatively and briefly discuss the stability of the steady state and the bifurcation of oscillating solutions in the framework furnished by equation (41a). Let us rewrite it retaining up to fifth order terms in powers of the amplitude:

$$\frac{da}{dt} = a \cdot \left(\kappa_0(\mu) + \kappa_2(\mu) \cdot a^2 + \kappa_4(\mu) \cdot a^4\right) \tag{58}$$

In (57) the analytical formulae (45) that relate the coefficients $\kappa_0, \kappa_2, \kappa_4$ with the parameters space of the mathematical model of the BWR are summarized by $\kappa_0(\mu)$, $\kappa_2(\mu)$ and $\kappa_4(\mu)$. Here μ represents either one subset of the above-mentioned parameters space (for example, the plane of the steady state reactor power and coolant flow) or only one parameter that varies while the others remain fixed (bifurcation parameter). Possible examples of a single bifurcation parameter are power, coolant flow or the parameter that relates excess void reactivity with excess fuel temperature.

The solution $a = 0$ always exists as a fixed point of the dynamics. It is locally stable for $\kappa_0(\mu) < 0$ and unstable for $\kappa_0(\mu) > 0$.

If $\kappa_0(\mu)$, $\kappa_2(\mu)$ and $\kappa_4(\mu)$ are all positive or all negative, $a = 0$ is the only fixed point of the dynamics. According to the truncated dynamics given by (57), $a = 0$ would be globally stable if all the coefficients are negative.

If the bi-quadratic equation $\kappa_0 + \kappa_2 \cdot a^2 + \kappa_4 \cdot a^4 = 0$ has real roots, each positive root corresponds to the amplitude of a possible limit cycle of oscillation.

To find the stability of the steady state, equation (43) must be solved first, to determine $\omega_0(\mu)$, the frequency of small amplitude oscillations around the steady state of the reactor (in this case the offset given by equation (33) is negligible). Then, $\omega_0(\mu)$ must be substituted in equation (44a) to determine $\kappa_0(\mu)$

In fact, what equation (43) determines is ${\omega_0}^2(\mu)$. If for the given combination of reactor parameters all the real roots of (43) are negative, the reactor can't lose its stability in an oscillatory way and the use of a KBM ansatz is inappropriate. So the region Ω_0 of parameters space Ω where (43) doesn't have a positive root must be excluded, and the rest of the points of parameters space must be considered as potential candidates to apply KBM.

As this averaging method presupposes time scales of different orders of magnitude, not all the points of $\Omega - \Omega_0$ are suitable to apply KBM.

From equation (54) we see that $\omega_0(\mu)$ tends to increase asymptotically like $\sqrt{P_0}$ so $\frac{{\omega_V}^2}{2 \cdot \omega_0}$ would also increase asymptotically like $\sqrt{P_0}$. Then, we can expect from equation (51) a change of sign in $\kappa_0(\mu)$ from negative to positive when the steady power increases enough and the other parameters remain fixed.

Being this way, the equation $\kappa_0(\mu) = 0$ determines the boundary of marginal stability and determines in parameters space two disjoint regions: one Ω_s of local stability of the steady regime of the reactor (if $\mu \in \Omega_s$ then $\kappa_0(\mu) < 0$), the other Ω_u of instability (if $\mu \in \Omega_u$ then $\kappa_0(\mu) > 0$).

Now, let us take the steady state power P_0 as variable parameter, leaving fixed the remaining parameters of the model. When P_0 is low enough, from (55a) we can expect that $\kappa_0(P_0) < 0$. But we can also expect a certain threshold of power $P_{0,u}$ such that $\kappa_0(P_{0,u}) = 0$. Then, for a higher steady state reactor power we have $\kappa_0(P_0) > 0$.

The condition $\kappa_0(P_0) = 0$ is equivalent to the equation:

$$\frac{h}{C} = \left(\frac{b_1 \cdot b_2 \cdot |\alpha_V|}{\Delta(\omega_0) \cdot C \cdot \Lambda} \right) \cdot P_0 + \frac{\beta}{\Lambda} \cdot \left(\left(\frac{h}{\omega_0 \cdot C} \right) \cdot \mathrm{I}_s(0) - \mathrm{I}_c(0) \right) \tag{59}$$

If $\omega_0(P_0)$ is known and is inserted in (59), then it should be possible to determine the threshold steady power $P_{0,u}$ as a function of the other parameters. This gives the boundary of stability: $\partial\Omega_*$

Crossing this boundary from the region characterized by $\kappa_0 < 0$ to the region with $\kappa_0 > 0$, the steady state regime given by $a = 0$ gets unstable. If $\kappa_0, \kappa_2, \kappa_4$ are all negative, the steady state is globally asymptotically stable.

If κ_2 and κ_4 remain negative when κ_0 crosses the stability boundary, a stable limit cycle appears when the steady state loses its stability (supercritical Poincaré-Andronov-Hopf (PAH) bifurcation).

We have a local bifurcation, in this case soft (PAH) excitation of direct loop oscillations.

It is possible to estimate the radius $\bar{a}_s$ of the stable limit cycle as a function of the model parameters μ, in the instability region ($\mu \in \Omega_u$) but near enough to $\partial\Omega_*$:

$$\bar{a}_s(\mu) \approx \sqrt{\frac{\kappa_0(\mu)}{|\kappa_2(\mu)|}} \tag{60a}$$

From (33) we find an offset corresponding to this stable cycle:

$$c_s(\mu) = -\log_e\left[I_0(\bar{a}_s(\mu))\right] \tag{60b}$$

According to the truncated dynamics given by equation (57), the possibility of having a Bautin's scenario, including a global bifurcation of limit cycles, appears for sets of parameters values in the stability region ($\mu \in \Omega_s$) near enough to the stability boundary, if $\kappa_0(\mu) < 0$, but simultaneously $\kappa_4(\mu) < 0$ and $\kappa_2(\mu) > 0$.

We would begin well inside the stability region Ω_s in parameters space, where the coefficients $\kappa_0(\mu), \kappa_2(\mu), \kappa_4(\mu)$ are negative and the steady state is globally asymptotically stable. As we as we approached the stability boundary $\partial\Omega_*$ where $\kappa_0(\mu)$ would change from negative to positive, it could happen that **before arriving there**, the discriminant function $D(\mu) = \kappa_2{}^2(\mu) - 4 \cdot \kappa_0(\mu) \cdot \kappa_4(\mu)$ changes from negative to positive, **being** $\kappa_2(\mu)$ **already positive** but $\kappa_4(\mu)$ being negative, as well as $\kappa_0(\mu)$.

Let us call $\tilde{\mu}$ a point in the region Ω_s of local stability in parameter space such that $\kappa_2^{\,2}(\tilde{\mu}) = 4 \cdot \kappa_0(\tilde{\mu}) \cdot \kappa_4(\tilde{\mu})$. For this first critical value there appears suddenly one finite oscillation amplitude:

$$\bar{a}(\tilde{\mu}) = +\sqrt{\frac{|\kappa_2(\tilde{\mu})|}{|\kappa_4(\tilde{\mu})|}} \tag{61}$$

The critical value $\tilde{\mu}$ belongs to a hyper-surface included in the stability region that separates, in the parameters space, the region of global stability from the region of local stability of the steady state of the reactor. Between this hyper-surface and the stability boundary, $D = \kappa_2^{\,2} - 4 \cdot \kappa_0 \cdot \kappa_4$ is less than $\kappa_2^{\,2}$.

If $\kappa_2 > 0$ there are two amplitudes of limit cycle:

$$\bar{a}_s(\mu) = +\sqrt{\frac{|\kappa_2| + \sqrt{\kappa_2^{\,2} - 4 \cdot \kappa_0 \cdot \kappa_4}}{2 \cdot |\kappa_4|}} \tag{62a}$$

$$\bar{a}_u(\mu) = +\sqrt{\frac{|\kappa_2| - \sqrt{\kappa_2^{\,2} - 4 \cdot \kappa_0 \cdot \kappa_4}}{2 \cdot |\kappa_4|}} \tag{62b}$$

For each point μ belonging to the region with only local stability: $\bar{a}_s(\mu) > \bar{a}_u(\mu)$

The upper branch $\bar{a}_s(\mu)$ corresponds to a stable limit cycle while the lower branch $\bar{a}_u(\mu)$ corresponds to an unstable limit cycle: a stable steady state surrounded by an unstable limit cycle surrounded by a stable limit cycle.

If all this were possible in the framework of the results obtained applying KBM, we would have entered in a regime of hard excitation of self-sustained oscillations. If the perturbation from steady state is not strong enough, the state of the reactor would return to the steady state after a transient characterized by decaying oscillations.

Only if the amplitude of the initial perturbation in the logarithmic power is greater than the threshold value $\bar{a}_u(\mu)$, the reactor would get unstable and its state variables would begin to oscillate approaching the stable oscillatory regime whose amplitude is given by Eq. (61a).

The corresponding angular frequency could be calculated by (truncating (41b) after the term of order four in the amplitude of oscillations):

$$\omega_s(\mu) \approx \omega_0(\mu) + \delta_2(\omega_0(\mu)) \cdot a_s^{\,2}(\mu) + \delta_4(\omega_0(\mu)) \cdot a_s^{\,4}(\mu) \tag{63}$$

The offset would be given by equation (33): $c_s(\mu) = -\log_e[I_0(\bar{a}_s(\mu))]$

When μ approaches to $\tilde{\mu}$ from the inside of the only local stability region, both branches coalesces in the amplitude of a half stable cycle given by Eq. (60): $\bar{a}_u(\tilde{\mu}) = \bar{a}_s(\tilde{\mu}) = \bar{a}(\tilde{\mu})$

If μ crosses to the global stability region, the possibility of having any limit cycle disappear.

When μ approaches to a value μ_u located on the stability boundary, where $\kappa_0(\mu_u) = 0$, the lower branch amplitude $\bar{a}_u(\mu)$ approaches to zero.

So, after having crossed the stability boundary, now for positive values of $\kappa_0(\mu)$, the only branch that would remain is the upper and stable one: a stable limit cycle surrounding an unstable steady state. We would have entered a regime of soft self excitation of sustained oscillations.
The whole bifurcation scenario involves both local and global bifurcations. In its entirety, and mainly when it is unexpected, this scenario is difficult to find with common bifurcation codes, because they work locally by locating the bifurcation points and reconstructing the adjacent parts of the branches.

4. CONCLUSIONS

To summarize, this contribution gives **only a first and incomplete analysis** to bifurcations and stability in a simplified model of a BWR, framed in an application of KBM method to derive approximate normal form equations for the amplitude and phase of the assumed oscillations and a relation between offset and amplitude.
However, a complete quantitative analysis should be done beginning with the analytical formulae derived in this paper for the generalized March-Leuba's model, and restricting the analysis to March-Leuba's ROM. For this model, realistic set of parameters were obtained adjusting parameters to the experimental results of stability tests done in BWR [9] [10] [17]. **Then, the assumptions made in the discussion of the present paper could be assessed, either to confirm or to rectify them.**

The Simplified March-Leuba's mathematical model of BWR dynamics was revisited and the transfer function that relates that relates excess void reactivity with excess fuel temperature was generalized.
A nonlinear differential integral equation for the logarithmic power was derived. This equation allows a description of the main characteristics related with stability and bifurcations. To apply KBM asymptotic method, an approximate formula was assumed for the logarithmic power: $x(t) \approx c(a(t)) + a(t) \cdot \cos\psi(t)$
Differential equations for $\frac{da}{dt}$ and $\frac{d\psi}{dt}$ were derived and solved, for each possible set of values of the parameters of the mathematical model.
Formula (33) was derived to relate the offset c with the amplitude a of oscillation.
Then, the nuclear reactor power $P(t)$ is given by the formula (57).
The solutions afforded by KBM method for the dimensionless logarithmic power describe oscillations with a fast frequency ω_0 modulated by a slow frequency $\frac{d\psi}{dt}(t) - \omega_0 = \frac{d}{dt}\theta(t)$ jointly with slowly varying amplitude, relative to a slowly varying offset $c(a(t))$.
The exponential dependence in (57) amplifies and reinforces the asymmetry of the oscillation in logarithmic power. **Strongly non-harmonic oscillations can be described by (57) even when the logarithmic power oscillates harmonically in a cycle with constant amplitude**

and stable frequency. This is a by-product of choosing the logarithmic power as a state variable.

If in certain subset of parameters space Ω equation (43) has no positive real roots, the KBM approach to stability and bifurcation can't be applied. So, this subset must be excluded.

An analytical formula for the boundary of marginal stability was derived in the multidimensional space of the parameters of the reactor. When the steady state crosses this hyper-surface, it presents an oscillatory lose of stability whose angular frequency may be calculated from positive real roots of equation (43).
The parameters that appear in the stability formulae are the steady state power, the neutron generation time, a core heat capacity, a heat transfer coefficient, a (squared) characteristic thermal frequency, a (squared) characteristic void frequency, and the parameters of a generalized kernel that gives a continuous delay effect of fuel temperature on the feedback void reactivity, including the residence time of steam bubbles in a typical channel and channel height, amongst others.

Because the residence time is proportional to the reciprocal of the coolant flow, and the frequency for small oscillations that appears in the equations that result from the application of KBM method to the integral- differential equation of logarithmic power, is proportional to the square root of the steady state power P_0, fixing the remaining parameters a stability diagram in the power–flow plane could be derived from the analytical formula giving the boundary of the region of local stability of the steady state of the reactor.

Besides the well-known global stability region, the results of applying the KBM method suggest that another region of local but not global stability appears that corresponds to a Bautin's bifurcation scenario. This behaviour was found and carefully studied during digital simulations done with the Paul Scherrer Institute-Technical University of Valencia-Technical University of Dresden ROM [5], [16].

The threshold power amplitude to instability could be calculated, as well as the amplitude and frequency of the stable limit cycle characteristic of hard self-excited oscillations, using the analytical formulae derived in the present paper, for any point in the region of local but not global stability of reactor's parameter space, and a suitable chosen impulse response function for void feedback reactivity.
Nevertheless, a complete stability study with a ROM must embrace all sets of possible parameter values, like the semi-analytical approach to bifurcation theory applied in reference [10], where PAH sub- and supercritical local bifurcations are studied, and the pertinent literature is reviewed.

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2013 International Nuclear Atlantic Conference - INAC 2013
Recife, PE, Brazil, November 24-29, 2013
ASSOCIAÇÃO BRASILEIRA DE ENERGIA NUCLEAR - ABEN
***ISBN*: 978-85-99141-05-2**

Non normal modal analysis of oscillations in boiling water reactors

Roberto Suarez-Antola[1,2] and Jose-Job Flores-Godoy[3]

[1]Dirección General de Secretaría, Ministerio de Industía, Energía y Minería,
Montevideo, Uruguay
roberto.suarez@miem.gub.uy

[2]Dpto. de Matemática, Facultad de Ingeniería y Tecnologías, Universidad Católica de Uruguay,
rsuarez@ucu.edu.uy

[3]Departamento de Física y Matemáticas, Universidad Iberoamericana,
México, D. F., México
job.flores@ibero.mx

ABSTRACT

The first objective of the present work is to construct a simple reduced order model for BWR stability analysis, combining a two nodes nodal model of the thermal hydraulics with a two modes modal model of the neutronics. Two coupled non-linear integral-differential equations are obtained, in terms of one global (in phase) and one local (out of phase) power amplitude, with direct and cross feedback reactivities given as functions of thermal hydraulics core variables (void fractions and temperatures). The second objective is to apply the effective life time approximation to further simplify the nonlinear equations. Linear approximations for the equations of the amplitudes of the global and regional modes are derived. The linearized equation for the amplitude of the global mode corresponds to a decoupled and damped harmonic oscillator. An analytical closed form formula for the damping coefficient, as a function of the parameters space of the BWR, is obtained. The coefficient changes its sign (with the corresponding modification in the decay ratio) when a stability boundary is crossed. This produces a supercritical Hopf bifurcation, with the steady state power of the reactor as the bifurcation parameter. However, the linearized equation for the amplitude of the regional mode corresponds always to an over-damped and always coupled (with the amplitude of the global mode) harmonic oscillator, for every set of possible values of core parameters (including the steady state power of the reactor) in the framework of the present mathematical model. The equation for the above mentioned over damped linear oscillator is closely connected with a non-normal operator. Due to this connection, there could be a significant transient growth of some solutions of the linear equation. This behavior allows a significant shrinking of the basin of attraction of the equilibrium state. The third objective is to apply the above approach to partially study the stability of the regional mode and some results that could be of some interest in relation with stability studies made with both, more complex and realistic reduced order models.

1. A SIMPLE NODAL-MODAL ROM

This simplified ROM consists in two coupled sub-models: the first describes neutron kinetics and second summarizes some aspects of the thermo-hydraulics. From now on t is time and $\vec{r}$ is a vector position in the (extrapolated) reactor core. Here neutron kinetics will be based in one group of neutrons (flux $\varphi(t,\vec{r})$, speed ν) and one group of delayed neutrons precursors (concentration $c(t,\vec{r})$, fraction of delayed neutron emitters β):

$$\frac{1}{\nu}\frac{\partial \phi}{\partial t} = H_0[\phi] + \delta H[\phi] - \beta\left(M_0[\phi] + \delta M[\phi]\right) + \lambda c \tag{1}$$

$$\frac{\partial c}{\partial t} = \beta\left(M_0[\phi] + \delta M[\phi]\right) - \lambda c \tag{2}$$

By definition: $H_0 = M_0 - L_0$ and $\delta H = \delta M - \delta L$, where $M_0(\vec{r})$ is the fission operator in the steady state, $\delta M(t, \vec{r})$ perturbation in the fission operator due to feedbacks loops; $L_0(\vec{r})$ is the loss operator in the steady state, $\delta L(t, \vec{r})$ is the perturbation in the loss operator.

1.1. Neutron kinetic modal model

The fundamental and first lambda modes $\varphi_0(\vec{r})$, $\varphi_1(\vec{r})$, respectively, are used for the spatial approximate representation of neutron flux:

$$\phi(t, \vec{r}) \approx (P_* + P_0(t))\varphi_0(\vec{r}) + P_1(t)\varphi_1(\vec{r}) \tag{3}$$

$$c(t, \vec{r}) \approx C_0(t)\varphi_0(\vec{r}) + C_1(t)\varphi_1(\vec{r}), \tag{4}$$

here P_* is the global (fundamental) mode amplitude that corresponds to the steady state power of the reactor, $P_0(t)$ is the perturbation of the global mode from its steady state and $P_1(t)$ is the amplitude of the regional (first azimuthal) mode. C_0 is the amplitude of the fundamental mode component for the delay neutron precusors and C_1 is the corresponding amplitude of the first mode. As usual, we neglect the contributions of the delayed neutron precursors to the feedback reactivities [1, 2, 3]. Then, from Eqs. (1–2), following a procedure similar to the one employed in reference [2, 3], it is possible to derive the following set of coupled ordinary differential equations to study the onset of global and regional power oscillations due to neutron-thermo-hydraulic coupling

$$\Lambda_0 \frac{dP_0(t)}{dt} = (\delta\rho_{00} - \beta)(P_* + P_0(t)) + \delta\rho_{01}P_1(t) + \Lambda_0\lambda C_0 \tag{5}$$

$$\frac{d}{dt}C_0 = \frac{\beta}{\Lambda_0}(P_* + P_0(t)) - \lambda C_0 \tag{6}$$

$$\Lambda_1 \frac{dP_1(t)}{dt} = \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s - \beta + \delta\rho_{11})P_1(t) + \Lambda_1\lambda C_1 \tag{7}$$

$$\frac{d}{dt}C_1 = \frac{\beta}{\Lambda_1}P_1(t) - \lambda C_1 \tag{8}$$

If $\widetilde{B}$ represents the extrapolated core

$$\Lambda_0 = \frac{1}{\nu}\frac{\int_{\widetilde{B}} \varphi_0\varphi_0 dV}{\int_{\widetilde{B}} \varphi_0 M_0[\varphi_0]\, dV}, \quad \Lambda_1 = \frac{1}{\nu}\frac{\int_{\widetilde{B}} \varphi_1\varphi_1 dV}{\int_{\widetilde{B}} \varphi_1 M_0[\varphi_1]\, dV} \tag{9}$$

are modal neutron generation times, $\rho_1^s = 1 - (1/k_1)$ is the subcriticality of the regional mode, and k_1 is the eigenvalue of the first (regional) lambda mode: $L_0[\varphi_1] = (1/k_1) M_0[\varphi_1]$. For the fundamental (global) lambda mode we assume criticality ($k_0 = 1$, so we have $L_0[\varphi_0] = M_0[\varphi_0]$). Direct reactivities $\delta\rho_{00}$, $\delta\rho_{11}$ and cross reactivities $\delta\rho_{01}$, $\delta\rho_{10}$ are given by the formulae:

$$\delta\rho_{mn} = \frac{\int_{\widetilde{B}} \varphi_m \delta H[\varphi_n]\, dV}{\int_{\widetilde{B}} \varphi_m M_0[\varphi_m]\, dV} \quad m, n = 0, 1 \tag{10}$$

In principle, reactivities can be written as the sum $\delta\rho_{mn} = \delta\rho^F_{mn} + \delta\rho^C_{mn} + \delta\rho_{\text{ext},m,n}$ of three contributions, one due to internal feedbacks loops due to temperature and void effects, $\delta\rho^F_{mn}$, other contribution $\delta\rho^C_{mn}$ from feedback loops due to the automatic control system, and another contribution $\delta\rho_{\text{ext},mn}$ due to external effects that can be described as additive terms appearing in the perturbation δL of the leak operator.

Now, let us suppose that the extrapolated core, is composed by two extrapolated half-core regions, a and b: $\widetilde{B} = \widetilde{B}_a \cup \widetilde{B}_b$. Furthermore, we assume that there is a *one-to-one* and *onto* **symmetry transformation** $\vec{r'} = \mathrm{T}(\vec{r})$ that maps $\widetilde{B}_a$ onto $\widetilde{B}_b$, and $\widetilde{B}_b$ onto $\widetilde{B}_a$, being: $\mathrm{T}\left(\vec{r'}\right) = \mathrm{T}(\mathrm{T}(\vec{r})) = \mathrm{T}^2(\vec{r}) = \vec{r}$. The fundamental mode is symmetric under $\mathrm{T} : \varphi_0(\mathrm{T}(\vec{r})) = \varphi_0(\vec{r})$ for every $\vec{r} \in \widetilde{B}$. The first mode is anti-symmetrical under $\mathrm{T} : \varphi_1(\mathrm{T}(\vec{r})) = -\varphi_1(\vec{r})$ for every $\vec{r} \in \widetilde{B}$. Now, $M_o[\cdot]$ is a linear scalar operator that has a spatial dependence: $M_o[\varphi] = \nu\Sigma_{f,0}(\vec{r})\varphi(t,\vec{r})$. We suppose that $\Sigma_{f,0}(\mathrm{T}(\vec{r})) = \Sigma_{f,0}(\vec{r})$ for every $\vec{r} \in \widetilde{B}$ (as consequence of a symmetrical fuel distribution). Then:

$$\int_{\widetilde{B}_a} \varphi_m M_0[\varphi_m]\, dV = \int_{\widetilde{B}_b} \varphi_m M_0[\varphi_m]\, dV = \frac{1}{2}\int_{\widetilde{B}} \varphi_m M_0[\varphi_m]\, dV \tag{11}$$

Taking these last equalities into account, it is possible to split the feedback reactivities as the sum of two terms, one corresponding to region a and the other to region b:

$$\delta\rho_{mn} = \frac{1}{2}\left(\frac{\int_{\widetilde{B}_a} \varphi_m \delta H[\varphi_n]\, dV}{\int_{\widetilde{B}_a} \varphi_m M_0[\varphi_m]\, dV} + \frac{\int_{\widetilde{B}_b} \varphi_m \delta H[\varphi_n]\, dV}{\int_{\widetilde{B}_b} \varphi_m M_0[\varphi_m]\, dV}\right) = \frac{1}{2}\left(\delta\rho^a_{mn} + \delta\rho^b_{mn}\right) \quad m,n = 0,1 \tag{12}$$

Now, let us introduce the symmetric $\delta H_s = \delta M_s - \delta L_s$ and the anti-symmetric $\delta H_a = \delta M_a - \delta L_a$ operators defined by $\delta H_s = (\delta H(\vec{r}) + \delta H(\mathrm{T}(\vec{r})))/2$ and $\delta H_a = (\delta H(\vec{r}) - \delta H(\mathrm{T}(\vec{r})))/2$. Due to the different symmetry of φ_0 and φ_1, it follows that:

$$\delta\rho_{ii} = \frac{\int_{\widetilde{B}} \varphi_i \delta H_s[\varphi_i]\, dV}{\int_{\widetilde{B}} \varphi_i M_0[\varphi_i]\, dV},\ i = 0,1 \tag{13}$$

$$\delta\rho_{ij} = \frac{\int_{\widetilde{B}} \varphi_i \delta H_a[\varphi_j]\, dV}{\int_{\widetilde{B}} \varphi_i M_0[\varphi_i]\, dV},\ i \neq j,\ i,j = 0,1 \tag{14}$$

The direct mode reactivities $\delta\rho_{00}$ and $\delta\rho_{11}$ involve the **symmetric operator** δH_s, while the cross mode reactivities $\delta\rho_{01}$ and $\delta\rho_{10}$ involve the **anti-symmetric operator** δH_a. It follows that in principle they could be all different.

Now, we need formulae for the direct ($\delta\rho_{00}$,$\delta\rho_{11}$) and cross reactivities ($\delta\rho_{01}$,$\delta\rho_{10}$) as functions of thermal-hydraulics core variables and control system variables. From Eq. (13) we see that the direct reactivities, due to the appeareance of δH_s in both formulae, must show a similar dependence of the thermo-hydraulic and control variables, although they are not proportional one to the other.

However, from Eq. (14) taking into account that $\langle \varphi_0, \delta H_a [\varphi_1] \rangle = \langle \varphi_1, \delta H_a [\varphi_0] \rangle$, we see that **the cross reactivities are proportional**

$$\delta\rho_{0,1} = f_{01}\delta\rho_{1,0} \quad \text{with} \quad f_{01} = \frac{\int\limits_{\widetilde{B}} \varphi_1 M_0 [\varphi_1] \, dV}{\int\limits_{\widetilde{B}} \varphi_0 M_0 [\varphi_0] \, dV} \tag{15}$$

The proportionality factor depends of the spatial distribution of the macroscopic fission cross sections of the fuel in the core and of the lambda modes. Due to the usual symmetric distribution of fuel in reactor core, and to the differences in the spatial fields of the fundamental and the first lambda modes, we can expect that the non-dimensional factor f_{01} be almost an order of magnitude less than 1. So the strength of the coupling between the modes is different. The regional mode couples weakly with the global mode, in comparison with the strength of the coupling of the global mode with the regional one. This must be taken into due account in the analysis of the modal-nodal model.

1.2. Thermo-hydraulic nodal model

As said before, the core B is split into two symmetric halves (nodes) B_a and B_b. A fuel temperature and a channel void fraction are assigned as state variables to each half core.

1.2.1. Fuel to coolant heat transfer

Introducing: C (fuel heat capacity), h (a fuel to coolant heat transfer coefficient), T_* (a steady state temperature), P_* (a steady state power), T_a, P_a (temperature and power assigned to B_a) and T_b, P_b (temperature and power assigned to B_b) we postulate these linear heat transfer equations:

$$\frac{C}{2}\frac{d\,(T_a - T_*)}{dt} = \left(P_a - \frac{P_*}{2}\right) - \frac{h}{2}\,(T_a - T_*) \tag{16}$$

$$\frac{C}{2}\frac{d\,(T_b - T_*)}{dt} = \left(P_b - \frac{P_*}{2}\right) - \frac{h}{2}\,(T_b - T_*) \tag{17}$$

If $T_0 = (T_a + T_b)/2$ is the **average temperature** and $P = P_a + P_b$ is the **instantaneous power** of the reactor, from Eqs. (16–17) it follows, if by definition $\delta T_0 = T_0 - T_*$:

$$C\frac{d\delta T_0}{dt} = P_0 - h\delta T_0 \tag{18}$$

If $\Delta T = (T_b - T_a)$ and $P_1 = P_b - P_a$ then, from Eqs. (16–17) we obtain:

$$C\frac{d\,(\Delta T)}{dt} = 2P_1 - h\Delta T \tag{19}$$

Introducing the operator:

$$\widehat{L}_T\,[f\,(t)] = \frac{df\,(t)}{dt} + \frac{h}{C}f\,(t) \tag{20}$$

Equations (18) and (19) may be recast as follows:

$$\widehat{L}_T\left[\delta T_0\left(t\right)\right] = \frac{1}{C}\left(P - P_*\right) = \frac{1}{C}P_0 \tag{21}$$

$$\widehat{L}_T\left[\Delta T\left(t\right)\right] = \frac{2}{C}P_1 \tag{22}$$

Here, by definition, $P_0 = P - P_*$ is **the perturbation** of the total core power relative to its steady state. $P_1 = P_b - P_a$, may be considered as a **perturbation in the reactor's symmetry** as expressed by the difference in the instantaneous power between the two halves of the core.

1.2.2. Steam void dynamics

Let us consider the void fractions α_a in B_a and α_b in B_b. The mean void fraction in the core is $\alpha \approx \left(\alpha_a + \alpha_b\right)/2$ and the deviation between half core void fractions is $\Delta\alpha \approx \left(\alpha_b - \alpha_a\right)$. If α_* is the void fraction in the symmetric steady state with reactor power P_*, we define the perturbations

$$\delta\alpha_a = \alpha_a - \alpha_* \quad \delta\alpha_b = \alpha_b - \alpha_* \quad \delta\alpha = \frac{1}{2}\left(\delta\alpha_a + \delta\alpha_b\right) \tag{23}$$

Following the classical March-Leuba's ROM, the connection between $\delta\alpha$ and δT_0 is frequently assumed to be of the type:

$$\frac{d^2\delta\alpha}{dt^2} + b_1\frac{d\delta\alpha}{dt} + b_2\delta\alpha = b_3\delta T_0 + b_4\frac{d\delta T_0}{dt} \tag{24}$$

The derivation of Eq. (24) as well as a set of analytical formulae for b_1, b_2, b_3 and b_4, in the case of the direct loop mechanism, is done substituting an exponential $e^{-s\tau}$ in Laplace's variable by its Padé's $(2,2)$ approximation. Now (as shown, for example, in the Appendix of [4]), it is possible to derive the following approximate formulae for the parameters of an analogous equation for the void feedback reactivity

$$b_1 \approx \frac{6}{\tau}, \quad b_2 \approx \frac{12}{\tau^2}, \quad b_3 \approx -\frac{6\Theta H^2}{\tau^2}, \quad b_4 \approx -\frac{\Theta H^2}{\tau} \tag{25}$$

Here $\tau \approx \frac{H}{\overline{V}_0}$ is an estimation of the residence time of steam bubbles in the channel, H is channel height, Θ is a positive parameter (that in our case has a different meaning that in [4])), and $\overline{V}_0$ is a mean void propagation velocity. Then, from b_1 and b_2 from Eq. (25) it follows that the natural frequency of the void oscillator is $\omega_0 = \sqrt{b_2} = \frac{\sqrt{12}}{\tau}$ and the friction factor $\zeta_0 = \frac{b_1}{2\omega_0} = \frac{3}{\sqrt{12}} < 1$. However, if a higher order Padé's approximation is used, or if a different mathematical model of void feedback is constructed, other equations relating excess void reactivity with excess fuel temperature are obtained, instead of Eq. (24). So let us generalize Eq. (24) for excess void reactivity, and from now on, let us work with a **generic impulse response function** $h_V\left(t\right)$ that induces a convolution operator

$$\mathcal{H}_V\left(q(t)\right) = \int_0^\infty h(u)q(t-u). \tag{26}$$

Then we write: $\delta\alpha_a(t) = G_{V,a}\mathcal{H}_{V,a}(\delta T_a(t))$, $\delta\alpha_b(t) = G_{V,b}\mathcal{H}_{V,b}(\delta T_b(t))$. Here $\delta T_a(t) = T_a(t) - T_*$ and $\delta T_b(t) = T_b(t) - T_*$ so $\delta T_0(t) = (\delta T_a(t) + \delta T_b(t))/2$. The static gain factors $G_{V,a}$, $G_{V,b}$ are introduced in order to have

$$\int_0^\infty h_{V,a}(u)\,du = \int_0^\infty h_{V,b}(u)\,du = 1. \tag{27}$$

If the two halves are symmetric enough: $h_{V,a}(u) = h_{V,b}(u) = h_V(u)$ and $G_{V,a} = G_{V,b} = G_V$, then:

$$\delta\alpha(t) = G_V\mathcal{H}_V(\delta T_0(t)) \tag{28}$$

$$\Delta\alpha(t) = G_V\mathcal{H}_V(\Delta T(t)) \tag{29}$$

The oscillatory impulse response function $h_V(t)$ is not suitable to define a useful mean void delay time t_V by means of the integral $\int_0^\infty t h_V(t)dt$ because, as a consequence of the changes of sign in $h_V(t)$, this integral underestimates the delay. However, as shown elsewhere [5], a suitable average time lag may be estimated using a damped exponential $E(t)$ that gives the decreasing amplitudes of the oscillations in the impulse response function $h_V(t)$: $E(t) = \mu e^{-\mu t}$. The positive normalizing factor μ is introduced in order to have $\int_0^\infty E(t)dt = 1$. Then an **average void delay time** can be estimated by the formula:

$$t_V \approx \theta\bar{t} = \theta\int_0^\infty tE(t)dt = \theta\frac{1}{\mu} \tag{30}$$

The non-dimensional factor θ can be taken as 1. When the relation between void fraction and fuel temperature is the classical one (Eq. (24)), $\mu = \frac{b_1}{2}$ and $t_V \approx \theta\frac{2}{b_1} \approx \frac{2}{b_1}$: the **time lag increases with decreasing damping** in void fraction oscillations.

1.3. Combination of the two-mode neutron kinetic model with the two-node thermo-hydraulic model through feedback reactivities

To connect the modal model of the neutronics with the nodal models for the thermo-hydraulics, let as introduce a temperature field and a void fraction field:

$$T(t,\vec{r}) = T_*\varphi_0(\vec{r}) + \delta T_0(t)\,\varphi_0(\vec{r}) + \Delta T(t)\,\varphi_1(\vec{r}) \tag{31}$$

$$\alpha(t,\vec{r}) = \alpha_*\varphi_0(\vec{r}) + \delta\alpha_0(t)\,\varphi_0(\vec{r}) + \Delta\alpha(t)\,\varphi_1(\vec{r}) \tag{32}$$

Here $T_*\varphi_0(\vec{r})$ and $\alpha_*\varphi_0(\vec{r})$ are certain temperature and void fraction profiles given in terms of the temperature T_* and void fraction α_* assigned to the steady state. Substituting Eqs. (31–32) in the evolution operator:

$$H(t,T,\alpha) = H_0(t,T_*\varphi_0,\alpha_*\varphi_0) + \delta H\Big(t,\delta T_0(t)\,\varphi_0 + \Delta T(t)\,\varphi_1,\delta\alpha_0(t)\,\varphi_0 + \Delta\alpha(t)\,\varphi_1\Big) \tag{33}$$

Developing in terms of the perturbations in the temperature and the void fields up to the linear terms with $\delta H = \delta H_s + \delta H_a$, we find:

$$\delta H_s[\cdot] = \delta T_0\widehat{S}_T[\cdot] + \delta\alpha_0\widehat{S}_V[\cdot] \tag{34}$$

$$\delta H_a[\cdot] = \delta T_0\widehat{A}_T[\cdot] + \delta\alpha_0\widehat{A}_V[\cdot] \tag{35}$$

In these relation $\widehat{S}$ and $\widehat{A}$ are symmetric and antisymmetric linear operators, respectively, corresponding to the effects on the production and loss operators due to temperature and void fraction variations. Given any operator $\widehat{O}$, we introduce the inner product notation:

$$\left\langle \varphi, \widehat{O}\left[\psi\right] \right\rangle = \int\limits_{\widetilde{B}} \phi \widehat{O}\left[\psi\right] dV \tag{36}$$

Then from Eqs. (13–14) and Eqs. (34–35) we obtain the following formulae for the direct and cross feedback reactivities. Direct, global mode:

$$\delta\rho_{00}^{F} = \frac{\left\langle \varphi_0, \delta H_s\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle} = c_{00}^{FT}\delta T_0 + c_{00}^{FV}\delta\alpha \tag{37}$$

The direct Doppler c_{00}^{FT} and void c_{00}^{FV} feedback coefficients for the global mode are given by:

$$c_{00}^{FT} = \frac{\left\langle \varphi_0, \widehat{S}_T\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle}, \quad c_{00}^{FV} = \frac{\left\langle \varphi_0, \widehat{S}_V\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle}. \tag{38}$$

Both c_{00}^{FT} and c_{00}^{FV} are negative. Direct, regional mode:

$$\delta\rho_{11}^{F} = \frac{\left\langle \phi_1, \delta H_s\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle} = c_{11}^{FT}\delta T_0 + c_{11}^{FV}\delta\alpha \tag{39}$$

The direct Doppler c_{11}^{FT} and void c_{11}^{FV} feedback coefficients for the regional mode are given by:

$$c_{11}^{FT} = \frac{\left\langle \varphi_1, \widehat{S}_T\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle}, \quad c_{11}^{FV} = \frac{\left\langle \varphi_1, \widehat{S}_V\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle}. \tag{40}$$

Cross terms, from regional to global:

$$\delta\rho_{01}^{F} = \frac{\left\langle \varphi_0, \delta H_a\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle} = c_{01}^{FT}\Delta T + c_{01}^{FV}\Delta\alpha \tag{41}$$

The cross Doppler c_{01}^{FT} and void c_{01}^{FV} feedback coefficients are given by:

$$c_{01}^{FT} = \frac{\left\langle \varphi_0, \widehat{A}_T\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle}, \quad c_{01}^{FV} = \frac{\left\langle \varphi_0, \widehat{A}_V\left[\varphi_1\right] \right\rangle}{\left\langle \varphi_0, M_0\left[\varphi_0\right] \right\rangle}. \tag{42}$$

Cross terms, from global to regional:

$$\delta\rho_{10}^{F} = \frac{\left\langle \varphi_1, \delta H_a\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle} = c_{10}^{FT}\Delta T + c_{10}^{FV}\Delta\alpha \tag{43}$$

The cross Doppler c_{10}^{FT} and void c_{10}^{FV} feedback coefficients are given by:

$$c_{10}^{FT} = \frac{\left\langle \varphi_1, \widehat{A}_T\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle}, \quad c_{10}^{FV} = \frac{\left\langle \varphi_1, \widehat{A}_V\left[\varphi_0\right] \right\rangle}{\left\langle \varphi_1, M_0\left[\varphi_1\right] \right\rangle}. \tag{44}$$

As consequence of the relation of proportionality Eq. (15) between cross feedback reactivities, we obtain for the cross Doppler and void feedback coefficients:

$$c_{01}^{FT} = f_{01} c_{10}^{FT}, \quad c_{01}^{FV} = f_{01} c_{10}^{FV}. \tag{45}$$

The void fractions $\delta\alpha$ and $\Delta\alpha$ are related with the temperatures $\delta T_0 = T_0(t) - T_*$ and ΔT by the equations Eqs. (28–29). So, we obtain the following formulae for direct and cross feedback reactivities in terms of temperature and power perturbations:

$$\begin{aligned} \delta\rho_{00}^F &\approx c_{00}^{FT} \delta T_0(t) + c_{00}^{FV} G_V \mathcal{H}_V(\delta T_0(t)) \\ \delta\rho_{11}^F &\approx c_{11}^{FT} \delta T_0(t) + c_{11}^{FV} G_V \mathcal{H}_V(\delta T_0(t)) \\ \delta\rho_{10}^F &\approx c_{10}^{FT} \Delta T + c_{10}^{FV} G_V \mathcal{H}_V(\Delta T(t)) \end{aligned} \tag{46}$$

with $\mathcal{H}_V(\cdot)$ defined in Eq. (26).

Eliminating C_0 and C_1 from Eqs. (5–8) and introducing the kernel $D(t) = \lambda e^{-\lambda t}$ and the convolution-like operator $\mathcal{D}(q(t)) = \int_0^\infty D(u)\,(q(t-u) - q(t))\,du$, we obtain the following nonlinear differential-integral equations

$$\Lambda_0 \frac{dP_0(t)}{dt} = \delta\rho_{00}(P_* + P_0(t)) + \delta\rho_{01} P_1(t) + \beta\,\mathcal{D}(P_0(t)) \tag{47}$$

$$\Lambda_1 \frac{dP_1(t)}{dt} = \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11}) P_1(t) + \beta\,\mathcal{D}(P_1(t)) \tag{48}$$

To derive the effective life time approximation, in the term that gives the effect of delayed neutron emitters in these equations we approximate the mode amplitudes up to linear terms in u. Neglecting higher order terms can be justified if the time scale of variation of the mode amplitudes is greater than the time scale of $D(t)$, that is $1/\lambda$, which is of the order of 10 s. Introducing the **effective life time scale** $\Lambda_e = \Lambda + \beta t_D$, being $t_D = \int_0^\infty D(u)\,u du = 1/\lambda$ the time constant of the delayed neutrons emitters, the equations of the modes are reduced to these ones:

$$\Lambda_e \frac{dP_0(t)}{dt} \approx \delta\rho_{00}(P_* + P_0(t)) + \delta\rho_{01} P_1(t) \tag{49}$$

$$\Lambda_e \frac{dP_1(t)}{dt} \approx \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11}) P_1(t) \tag{50}$$

The first objective of the present work has been obtained with Eqs. (5–8) or equivalently Eqs. (47–48), jointly with Eqs. (46), (18–19), (28–29).

2. DAMPED HARMONIC OSCILLATORS FOR GLOBAL AND REGIONAL MODES: SLOW PROCESSES APPROXIMATION

To study stability and bifurcation with analytical technics it convenient to obtain first oscillator equations for the global and regional mode amplitudes. Linearizing Eqs. (49–50) we obtain

$$\Lambda_e \frac{dP_0(t)}{dt} \approx \delta\rho_{00} P_* \tag{51}$$

$$\Lambda_e \frac{dP_1(t)}{dt} \approx \delta\rho_{10} P_* + \rho_1^s P_1(t) \tag{52}$$

Applying the heat transfer operator $\widehat{L}_T$ to Eqs. (51–52) with Eq. (46) and Eqs. (21–22) we obtain

$$\frac{d^2P_0(t)}{dt^2} + \frac{h}{C}\frac{dP_0(t)}{dt} + \omega_{gTe}^2 P_0(t) + \omega_{gVe}^2 \mathcal{H}_V(P_0(t)) \approx 0 \tag{53}$$

$$\frac{d^2P_1(t)}{dt^2} + \left(\frac{h}{C} + \frac{|\rho_1^s|}{\Lambda_e}\right)\frac{dP_1(t)}{dt} + \left(\frac{|\rho_1^s|}{\Lambda_e}\frac{h}{C} + \omega_{rTe}^2\right)P_1(t) + \omega_{rVe}^2 \mathcal{H}_V(P_1(t)) \approx 0 \tag{54}$$

with

$$\omega_{gTe} = \sqrt{\frac{|c_{00}^{FT}|\,P_*}{\Lambda_e C}},\ \omega_{gVe} = \sqrt{\frac{|c_{00}^{FV}|\,G_V P_*}{\Lambda_e C}},\ \omega_{rTe} = \sqrt{2\frac{|c_{10}^{FT}|\,P_*}{\Lambda_e C}},\ \omega_{rVe} = \sqrt{2\frac{|c_{10}^{FV}|\,G_V P_*}{\Lambda_e C}}$$

are certain thermal and void frequencies of the uncoupled global mode and the coupled regional mode in the effective life time approximation, respectively; assuming that c_{00}^{FT}, c_{00}^{FV}, c_{10}^{FT} and c_{10}^{FV} are negative and $\mathcal{H}_V(\cdot)$ from Eq. (26). Observe that for slow enough process, the convolution operator can be approximates as $\mathcal{H}_V(f(t)) \approx f(t) - t_V\frac{f(t)}{dt}$. Using this approximation in Eqs. (53–54) we obtain

$$\frac{d^2P_0(t)}{dt^2} + 2\zeta_{ge}\omega_{ge}\frac{dP_0(t)}{dt} + \omega_{ge}^2 P_0(t) \approx 0 \tag{55}$$

$$\frac{d^2P_1(t)}{dt^2} + 2\zeta_{re}\omega_{re}\frac{dP_1(t)}{dt} + \omega_{re}^2 P_1(t) \approx 0 \tag{56}$$

By definition:

$$\omega_{ge} = \sqrt{\omega_{gTe}^2 + \omega_{gVe}^2},\quad 2\zeta_{ge}\omega_{ge} = \frac{h}{C} - \frac{\left|c_{00}^{FV}\right|G_V}{\Lambda_e C}t_V P_* \tag{57}$$

$$\omega_{re}^2 = \frac{h}{C}\frac{|\rho_1^s|}{\Lambda_e} + \omega_{rTe}^2 + \omega_{rVe}^2,\quad 2\zeta_{re}\omega_{re} = \frac{h}{C} + \frac{|\rho_1^s|}{\Lambda_e} - 2\frac{\left|c_{10}^{FV}\right|G_V}{\Lambda_e C}t_V P_* \tag{58}$$

So, there is negative damping (instability) when ζ_{ge} or ζ_{re} are negative. For each mode there is a stability boundary in the BWR parameters space, these boundaries can be found when $\zeta_{ge} \equiv 0$ or $\zeta_{re} \equiv 0$. The second objective of the present paper is fulfilled.

3. NON NORMAL BEHAVIOR OF THE REGIONAL MODE

The effective lifetime approximation is use to describe slow processes, i.e. the oscillations presented due to misfits of the control system. To study faster process the prompt-jump approximation can be used as an intermediate stage between the effective lifetime approximation and the complete model constructed in section 1.

3.1. The equation for the regional mode in the prompt-jump approximation: linearization, uncoupling and non-normal stability

Applying a prompt jump approximation to Eq. (7), linearizing the resulting equation we arrive to

$$(\beta + |\rho_1^s|)\frac{dP_1}{dt} - \lambda(-|\rho_1^s|)P_1 = \left(\frac{d\delta\rho_{10}^F}{dt} + \lambda\delta\rho_{10}\right)P_* \tag{59}$$

using the thermal operator from Eq. (20)

$$\left(\ddot{P}_1(t)+\frac{h}{C}\dot{P}_1(t)\right)+\frac{\lambda\left|\rho_1^s\right|}{(\beta+\left|\rho_1^s\right|)}\left(\dot{P}_1(t)+\frac{h}{C}P_1\right)=\frac{\widehat{L}_T\left[\delta\dot{\rho}_{10}^F\right]+\lambda\widehat{L}_T\left[\delta\rho_{10}^F\right]}{(\beta+\left|\rho_1^s\right|)}P_* \tag{60}$$

If we uncouple the global mode from the regional mode on Eq. (60) we obtain:

$$\frac{d^2P_1(t)}{dt^2}+\left(\frac{h}{C}+\frac{\lambda\left|\rho_1^s\right|}{\beta+\left|\rho_1^s\right|}\right)\frac{dP_1(t)}{dt}+\left(\frac{h}{C}\frac{\lambda\left|\rho_1^s\right|}{\beta+\left|\rho_1^s\right|}\right)P_1(t)=0 \tag{61}$$

Now, suppose that: (a) the reactor is in its steady state for every $t<0$, i.e., $P_1(t)=0$ and $\dot{P}_1(t)=0$ for every $t<0$, and (b) a sudden perturbation $\dot{P}_1\left(0^+\right)$ in the velocity of the regional mode P_1 appears at $t=0$. If we put $P_1(t)=u_1$, $a=h/C$ and $c=\left(\lambda\left|\rho_1^s\right|\right)/\left(\beta+\left|\rho_1^s\right|\right)$, we can rewrite Eq. (61) as

$$\frac{d^2u(t)}{dt^2}+(a+c)\frac{du_1(t)}{dt}+acu_1(t)=0. \tag{62}$$

We introduce a new variable defined by $u_2=\frac{du_1}{dt}+au_1$, then $\frac{du_2}{dt}=-cu_2$. Now we can write Eq. (62) as an equivalent linear system

$$\frac{d}{dt}\vec{u}=A\vec{u},\quad A=\begin{bmatrix}-a & 1\\ 0 & -c\end{bmatrix},\quad \vec{u}=\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{63}$$

The A matrix is a non-normal operator, so if both a and c are small enough relative to 1, we can expect a significant linear growth for certain solutions [6]. The solution of the equation, without forcing terms due to mode coupling and **non-linear** terms, with these initial conditions is:

$$P_1(t)=\frac{\dot{P}_1\left(0^+\right)}{a-c}\left(e^{-ct}-e^{-at}\right),\ a\neq c \tag{64}$$

If the time evolution of the regional mode amplitude $P_1(t)$ follows Eq. (64), then $|P_1(t)|$ reaches a maximum value at the instant:

$$t_*=\frac{1}{a-c}\log_e\left(\frac{a}{c}\right) \tag{65}$$

This maximum is given by

$$P_1\left(t_*\right)=\frac{\dot{P}_1\left(0^+\right)}{a-c}\left[\left(\frac{c}{a}\right)^{\frac{c}{a-c}}-\left(\frac{c}{a}\right)^{\frac{a}{a-c}}\right] \tag{66}$$

As an example, let us suppose that $a=\epsilon$ and $c=2\epsilon$. In this case $P_1\left(t_*\right)=\left[\dot{P}_1\left(0^+\right)\right]/(4\epsilon)$, so for a given $\dot{P}_1\left(0^+\right)$, if ϵ tends to zero $\left|P_1\left(0^+\right)\right|$ tends to infinite. A consequence of this, in the non-linear regime, could be a meeting of non-normality with the nonlinearity as has already been observed in fluid mechanics [6].

3.2. Numerical simulation with the complete model

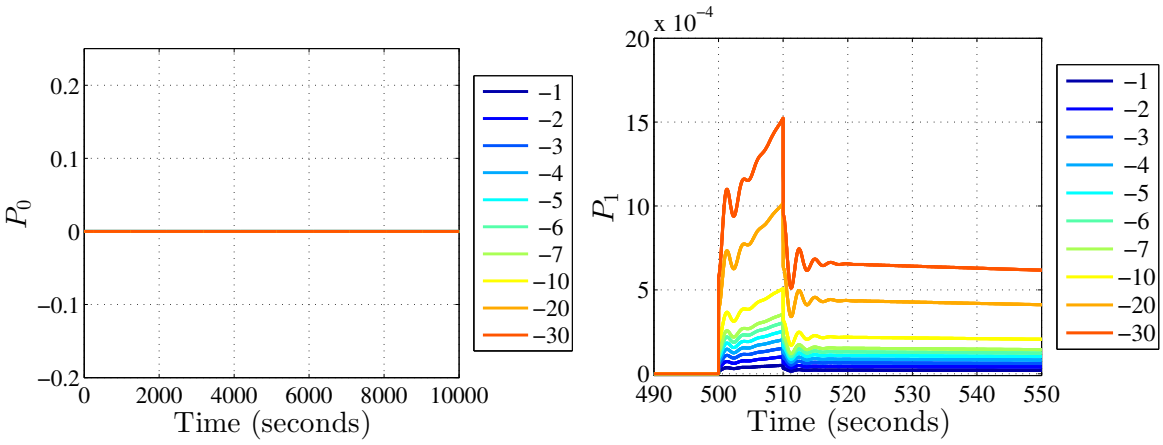
We chose a set of parameter values to exemplify stability issues related with the non normal behavior of the regional mode. $\Lambda_0=\Lambda_1=4\mathrm{e}-5[\mathrm{s}^{-1}]$, $\lambda=0.08[\mathrm{s}^{-1}]$, $\beta=$

0.0056, $\rho_1^s = -0.0001$, $h = 0.0092[\mathrm{JK^{-1}s^{-1}}]$, $C = 0.03994[\mathrm{JK^{-1}}]$, $c_{00}^{FT} = -1\mathrm{e}-5[\mathrm{K^{-1}}]$, $c_{00}^{FV} = -38\mathrm{e}-5$, $c_{11}^{FT} = -1.5680\mathrm{e}-5[\mathrm{K^{-1}}]$, $c_{11}^{FV} = -57.6\mathrm{e}-5$, $c_{01}^{FT} = -0.2\mathrm{e}-7[\mathrm{K^{-1}}]$, $c_{01}^{FV} = -0.2\mathrm{e}-7$, $c_{10}^{FT} = -1\mathrm{e}-7[\mathrm{K^{-1}}]$, $c_{10}^{FV} = -1\mathrm{e}-7$, $b_1 = 1[\mathrm{s^{-1}}]$, $b_2 = 6.28[(\mathrm{rad})\mathrm{s^{-2}}]$, $G_V = 0.495$.

The simulations started at the steady state of the reactor and lasted 20000 seconds. The simulations were made using Simulink from MATLAB R2012a using the solver `ode15s` with a relative tolerance of $1e-10$. We perturbed the reactor by a pulse of temperature difference in the reactor halves. The pulse width is of 10 s and the amplitude of the pulse was varied between -1°C and -30°C. Two case were studied depending on the value of the steady state power of the reactor P_*

3.2.1. Case I, $P_* = 1.1$

Observe that in Figure 1(a) the global mode remains at its steady state regardless of the perturbation used. Observed that the perturbation excites the regional mode only and the response of the regional mode is very small (see Figure 1(b)). From Figure 1(b) we can observed that there are three different time scales in the response of the amplitude of the regional mode; the fastest time constant is due to the contribution of Λ_0. The second fastest corresponds to the prompt-jump approximation. The slowest time constant correspond to an exponential decay toward the steady state of the regional amplitude. The slowest time constant is the same for all the perturbations.



(a) Global mode amplitude.

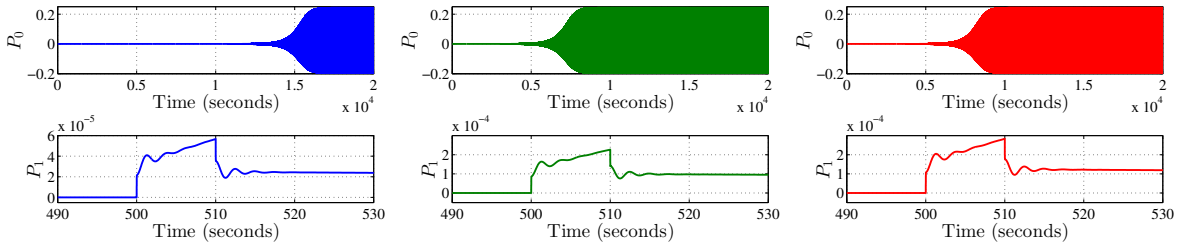
(b) Regional mode amplitude.

Figure 1: Reactor steady state stable under small perturbations with $P_* = 1.1$

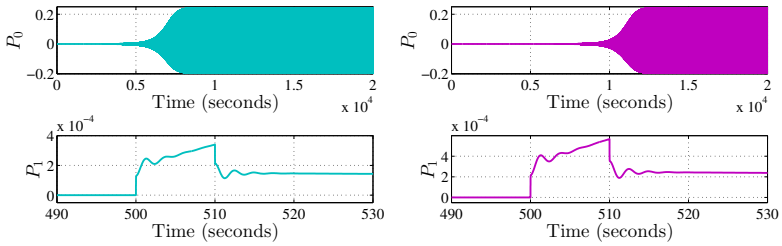
3.2.2. Case 2, $P_* = 1.229$

In this case we found three different scenarios depending on the magnitude of the temperature perturbation. When the amplitude of the perturbation was -3, -20, -30 the solution of the global mode after a very small intimal perturbation returns to its steady state. When the amplitude of the perturbation was -2, -7 the solution for global mode presents an initial small perturbation and converges to a small limit cycle. The amplitude

of this limit cycle is large enough to not be considered numerical noise. However for practical purposes this oscillation can be neglected. When the amplitude of the perturbation was -1, -4, -5, -6, -10 a significant limit cycle is excited in the global mode amplitude and all the cases the regional mode converges back to its steady state as can be seen in Figure 2. In Figure 3(a) we show the time lag for the limit cycle to initiate as a function of



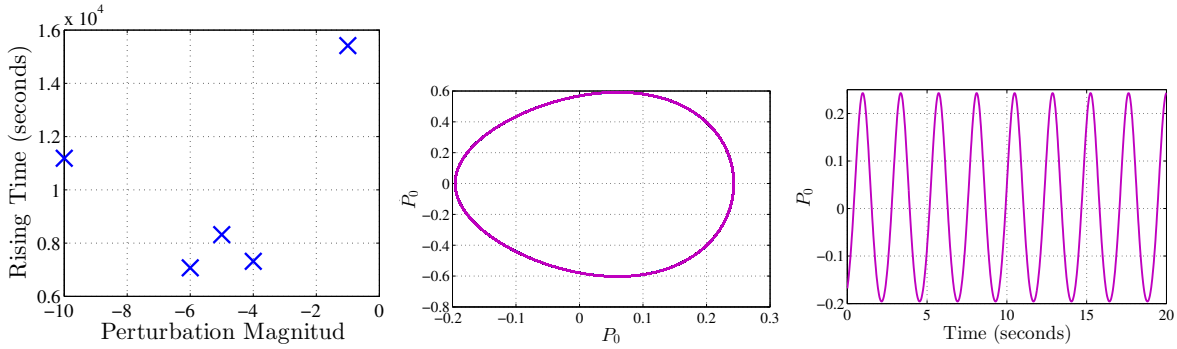
(a) Perturbation amplitude -1. (b) Perturbation amplitude -4. (c) Perturbation amplitude -5.



(d) Perturbation amplitude -6. (e) Perturbation amplitude -10.

Figure 2: Reactor steady state oscillatory under small perturbations with $P_* = 1.229$

the amplitude of the asymmetric perturbation in the core temperature. Observe that the time lag is not monotonically behaved as a function of the perturbation. In Figure 3(b) we can see the superposition of all established solutions: the superposition produces a unique closed orbit. In Figure 3(c) an example of the time series of the oscillation.



(a) Time lag to limit cycle (b) Closed orbit of limit cycle (c) Time series of limit cycle

Figure 3: Limit cycle properties

With these analysis and simulations results we achieved the third objective of the present work.

4. CONCLUSION

We present a new reduced order mathematical model of BWR dynamics based on symmetry considerations inspired by [7]. This model remains to be fully studied.

Using the effective life time approximation with suitable linear approximations and decoupling, we found an oscillatory loss of stability when the parameters of the reactor cross a stability boundary both for the global and regional modes. We obtained an analytical formulae for the corresponding decay rations and the oscillation frequencies near the stability boundary.

Using the prompt jump approximation we studied the possibility of non normal behavior of the regional mode. We chose a set of parameter to exemplify through numerical simulations the possible effects of the non normal behavior of the regional mode. For simulation purposes we use same set of parameter values for a wide range of steady state powers and long simulation times.

We found that for very mild perturbations in the temperature difference between the two reactor halves it is possible for the regional mode to excite into an oscillatory regime the global mode. Even though the magnitude of the regional mode response to the perturbation is very small, given enough time, the global mode oscillates. This happens for the steady state reactor power beyond a threshold. Also, there is evidence that there exist according to the model a very small oscillation excited for certain perturbation amplitudes and for other amplitudes the global mode returns to the steady state for the duration of the simulation.

For future work a suitable calibration of the parameters is needed in order to reproduce a realistic reactor dynamics. To further improve the span of this model, a first step would be to include a thermal hydraulic indirected loop to be able to study less idealized regional mode oscillations. It would also be necessary to include additional effects such as poison dynamics and heat produced by delayed neutron emitters amongst others in order to be able to extend the length of time in which the model predicts a working reactor behavior.

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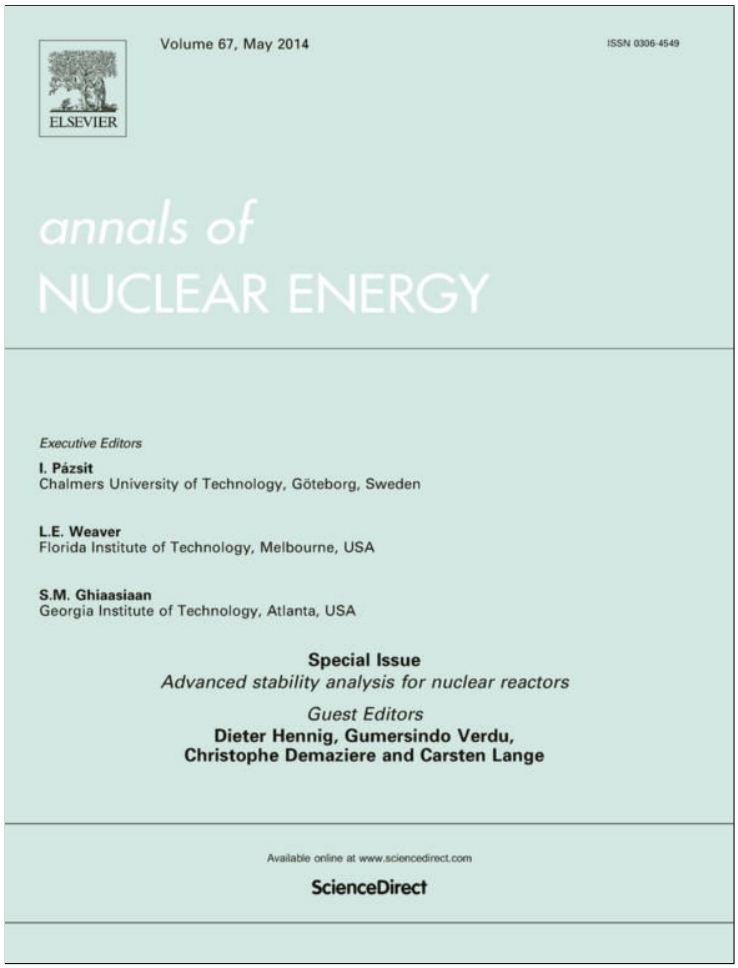

Annals of Nuclear Energy 67 (2014) 129–145

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Contribution to BWR stability analysis Part I: Analytical approach using a reduced order model

Roberto Suárez-Ántola [a,b,*], José-Job Flores-Godoy [c]

[a] Departamento de Matemáticas, Facultad de Ingeniería y Tecnologías, Universidad Católica del Uruguay Damaso A. Larrañaga, Montevideo, Uruguay
[b] Dirección General de Secretaría, Ministerio de Industía, Energía y Minería, Montevideo, Uruguay
[c] Departamento de Física y Matemáticas, Universidad Iberoamericana, México, DF, Mexico

ARTICLE INFO

Article history:
Received 30 May 2013
Accepted 11 November 2013
Available online 7 December 2013

Keywords:
Boiling water reactors
Stability theory
In-phase and out-of-phase oscillations
Reduced order models
Mathematical modeling

ABSTRACT

In this first part, we justify the use of a simple nonlinear set of ordinary integral differential equation to study the onset of global and regional power oscillations due to neutron-thermo-hydraulic coupling and possible malfunctions of the automatic control system of a boiling water reactor. This approach is framed in the construction of a new nodal-modal reduced order model of the core. Symmetry considerations are applied in the derivation of analytical formulae for direct and cross reactivities, including thermo-hydraulic and automatic control terms. Local bifurcations from the steady state of the reactor are studied from an analytical point of view, using both prompt-jump and effective life time approximation. Both thermal–hydraulic and automatic control system feedbacks are included. Asymptotic methods are used in the derivation of closed form analytical formulae of the stability boundaries in the space of reactor parameters as well as for amplitudes and frequencies of global and regional power oscillations. Besides analytical formulae for limit cycle oscillations are obtained.

1. Introduction

A full scale realistic mathematical modeling of a given Nuclear Power Plant (NPP), with focus in the Boiling Water Reactor (BWR), must be done beginning with a suitable coupled system of nonlinear partial differential equations. Then, the corresponding computational code for digital simulation of NPP steady states and transients is constructed and tested.

But fortunately, these NPPs equations are dissipative nonlinear partial differential equations (DNPDE). Many theoretical and computational studies have evidenced remarkable similarities between the long-time evolution of the manifold of solutions of this kind of DNPDE and the manifold of solutions of certain ordinary differential equations (ODE). The origin of this similarity in the asymptotic behavior of DNPDE and ODE is due to the following fact: asymptotic evolution of the solutions of DNPDE occurs near and exponentially approaching to a finite dimensional manifold, the so called inertial manifold.

An inertial manifold of solutions of a partial differential equation is a finite dimensional, compact and invariant set that attracts exponentially all trajectories that begin in a certain neighborhood of the manifold and is asymptotically complete in the following sense: For each trajectory that begins in a neighborhood of the inertial manifold, there is a trajectory included in the inertial manifold such that the distance between these trajectories tends to zero exponentially (Constantin et al., 1989; Malinietski, 2005).

Now, this inertial manifold corresponds to or is very well approximated by the solutions of a suitable set of nonlinear ordinary differential equations. With the inertial manifold theory, it is shown that certain dissipative PDEs have the same asymptotic behavior of an appropriate finite dimensional ODEs system.

Furthermore, the exponential approach of the solutions of the DNPDs to the corresponding inertial manifold (solutions of the asymptotically equivalent ODE) is in many cases fast enough to allow the dynamic study to be done entirely on the inertial manifold, that is, using ODE, reducing computational effort compared with traditional dynamic reduction procedures. This is the mathematical background that justifies the introduction of reduced order models (ROM) in studies of NPP dynamics and control: most often the transients are slow enough to allow the use of ODE instead of DNPDE in stability studies.

The use of reduced-order models (ROM) for the nonlinear stability analysis of BWR allows the application of a suitable combination of semi-analytical bifurcation theory with digital simulation of the dynamics, using well studied software for ordinary differential equations. The results thus obtained can be used to guide the numerical simulation using full system codes.

* Corresponding author at: Departamento de Matemáticas, Facultad de Ingeniería y Tecnologías, Universidad Católica del Uruguay Damaso A. Larrañaga, Montevideo, Uruguay.

E-mail addresses: rsuarez@ucu.edu.uy (R. Suárez-Ántola), job.flores@ibero.mx (J.-J. Flores-Godoy).

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http://dx.doi.org/10.1016/j.anucene.2013.11.017

Now, a typical ROM for BWR stability analysis may have a number of state variables near 20, interrelated by a corresponding system of nonlinear ordinary differential equations. However, simplified ROM construction, intended to describe reactor dynamics in certain time scales only, could be successfully undertaken since the first designs of generation I reactors, due to the hierarchy of widely separated time scales characteristic of nuclear reactor dynamics. This goes from prompt neutron-dominated effects (hundredths of seconds), to heat transfer from fuel to coolant (tenths of seconds), coolant transit times through core and precursor dominated effects (seconds), coolant transit time through the entire primary circuit (tens of seconds), oscillatory instabilities originated in automatic control systems (minutes), diurnal electric load variations and xenon flux tilting (tens of hours), samarium production (months), and fuel burn-up and transuranic isotope production (years). When the variables of interest (variables of reference) belong to a certain time scale, it is possible to simplify the dynamic analysis of nuclear reactors applying the following two principles to link scales of different orders of magnitude.

1. The variables belonging to processes with time scales at least an order of magnitude greater than the reference time scale, can be considered as frozen.
2. The variables belonging to processes that evolve with time scales at least an order of magnitude smaller than the variables of reference, after a short transient (produced in the so called inner time scale) can be considered as relaxed to equilibrium with these variables (evolving in the so called outer time scale).

As consequence, simpler models like the classical March-Leuba's ROM (March-Leuba et al., 1986) for in phase oscillations, Farawila and Pruitts ROM for in phase and out of phase oscillations (Farawila and Pruitt, 2006a,b) Turso et al. (1997) ROM for out of phase instabilities (coupled with the frequency domain LAPUR code for BWR stability analysis to simulate feedback) and the more complex model due to Muñoz-Cobo et al. (1996), in spite that they have a fewer number of state variables, reproduce very well some of the main characteristic of reactor's dynamics and are easier to study. Besides, there are efforts to develop simplified models from others points of view (Berman et al., 2012).

When the ROM is simple enough, it is sometimes possible to study local bifurcations of steady states and limit cycles and even some global bifurcations, using approximate analytical methods. The results thus obtained usually offer a useful guide to deeper stability and bifurcation studies using the same model or more complex ROM's of the same physical system (Rizwan-Uddin, 2006; March-Leuba and Rey, 1993; Muñoz-Cobo and Verdú, 1991).

The purpose of this first part of the paper is threefold:

1. To construct a simple ROM to study the onset of global and regional power oscillations due to neutron-thermo-hydraulic coupling through the direct loop, and possible malfunctions of the automatic control system.
2. To study from an analytical point of view the local bifurcations from the steady state of the reactor, using both prompt-jump and effective life time approximations.
3. To explore the possibility of applying asymptotic methods in the derivation of closed form analytical formulae of the stability boundaries in the space of reactor parameters as well as for amplitudes and frequencies of global and regional power oscillations.

2. A simple nodal-modal ROM

This simplified ROM consists in three coupled sub-models: the first describes neutron kinetics, the second summarizes some aspects of the thermo-hydraulics, and the third describes an automatic control system. From now on t is time and $\vec{r}$ is a vector position in the (extrapolated) reactor core.

Here neutron kinetics will be based in one group of neutrons (flux $\varphi(t,\vec{r})$, speed v) and one group of delayed neutrons precursors (concentration $c(t,\vec{r})$, fraction of delayed neutron emitters β):

$$\frac{1}{v}\frac{\partial\phi}{\partial t} = H_0[\phi] + \delta H[\phi] - \beta(M_0[\phi] + \delta M[\phi]) + \lambda c \tag{1}$$

$$\frac{\partial c}{\partial t} = \beta(M_0[\phi] + \delta M[\phi]) - \lambda c \tag{2}$$

By definition:

$$H_0 = M_0 - L_0$$
$$\delta H = \delta M - \delta L$$

where $M_0(\vec{r})$ is the fission operator in the steady state, $\delta M(t,\vec{r})$ is the perturbation in the fission operator due to feedbacks loops; $L_0(\vec{r})$ is the loss operator in the steady state, and $\delta L(t,\vec{r})$ is the perturbation in the loss operator.

Eliminating $c(t,\vec{r})$ and introducing the kernel $D(t) = \lambda e^{-\lambda t}$ it is fairly straightforward to derive the nonlinear partial differential-integral equation:

$$\frac{1}{v}\frac{\partial\phi(t,\vec{r})}{\partial t} = H_0[\phi](t,\vec{r}) + \delta H[\phi](t,\vec{r}) + \beta\int_0^\infty D(u)(\mathcal{M}[\phi](t-u,\vec{r}) - \mathcal{M}[\phi](t,\vec{r}))du \tag{3}$$

with $\mathcal{M}[\phi](t,\vec{r}) = M_0[\phi](t,\vec{r}) + \delta M[\phi](t,\vec{r})$

2.1. Neutron kinetic modal model

The fundamental and first lambda modes $\varphi_0(\vec{r})$, $\varphi_1(\vec{r})$, respectively, are used for the spatial approximate representation of neutron flux:

$$\phi(t,\vec{r}) \approx (P_* + P_0(t))\varphi_0(\vec{r}) + P_1(t)\varphi_1(\vec{r}) \tag{4}$$

here P_* is the global (fundamental) mode amplitude that corresponds to the steady state power of the reactor, $P_0(t)$ is the perturbation of the global mode from its steady state and $P_1(t)$ is the amplitude of the regional (first azimuthal) mode. Although the fundamental mode represents the spatial distribution of the neutron flux accurately at the critical state of the reactor, the first lambda modes are used in nuclear engineering for dynamic studies in a close neighborhood of the steady state of the reactor. The alternative is to use alpha modes but the difference between the corresponding alpha an lambda modes due to kinetic distortion is relatively small for the fundamental and first lambda mode (Bell and Glasstone, 1970; Akcasu et al., 1971; Hetrick, 1971; Stacey, 2007; Cao, 2008).

If ε is the energy liberated per fission, $\Sigma_f(t,\vec{r})$ is the field of macroscopic fission cross sections, and B is the region occupied by the reactor's core, then for one group of neutrons the prompt reactor power (neutron power) associated with the instantaneous fission rate can be estimated, as usual, by

$$P(t) \approx \int_B \varepsilon\Sigma_f(t,\vec{r})\phi(t,\vec{r})dV$$

In terms of the fundamental and first azimuthal lambda modes the neutron power is given by

$$P(t) \approx (P_* + P_0(t)) \int_B \varepsilon\Sigma_f(t,\vec{r})\varphi_0(\vec{r})dV + P_1(t) \int_B \varepsilon\Sigma_f(t,\vec{r})\varphi_1(\vec{r})dV$$

The first summand is neutron power related with the fundamental mode and the second one is neutron power related with the first azimuthal mode. If $\Sigma_f(t,\vec{r})$ is approximated by a steady state field $\Sigma_{f,0}(\vec{r})$, then the integrals over the core region are constant and the mode amplitudes $P_* + P_0(t)$, $P_1(t)$ can be interpreted as modal fission power amplitudes.

Besides the aforementioned prompt component, the thermal power deposited in the fuel and delivered to the coolant has another, delayed component, due to the decay heat of fission products. During transient nuclear reactor dynamic calculations that embrace relatively long time intervals, neglecting the delayed component of the thermal power could lead to an under-estimation of fuel temperatures with an under-estimation of Doppler feedback effects and errors in the calculation of prompt thermal power component. However, in the framework of the present model the considered transients last only minutes, so the variation in the delayed thermal power can be neglected.

As usual, we neglect the contributions of the delayed neutron precursors to the feedback reactivities (Lange, 2009; Lange et al., 2011). Then, from (3) and (4), following a procedure similar to the one employed in Lange (2009) and Lange et al. (2011), it is possible to derive the following set of coupled ordinary differential-integral equations to study the onset of global and regional power oscillations due to neutron-thermo-hydraulic coupling:

$$\Lambda_0 \frac{dP_0(t)}{dt} = \delta\rho_{00}(P_* + P_0(t)) + \delta\rho_{01}P_1(t) + \beta\int_0^\infty D(u)(P_0(t-u) - P_0(t))du \tag{5}$$

$$\Lambda_1 \frac{dP_1(t)}{dt} = \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11})P_1(t) + \beta\int_0^\infty D(u)(P_1(t-u) - P_1(t))du \tag{6}$$

If $\widetilde{B}$ represents the extrapolated core

$$\Lambda_0 = \frac{1}{v}\frac{\int_{\widetilde{B}}\varphi_0\varphi_0 dV}{\int_{\widetilde{B}}\varphi_0 M_0[\varphi_0]dV}, \quad \Lambda_1 = \frac{1}{v}\frac{\int_{\widetilde{B}}\varphi_1\varphi_1 dV}{\int_{\widetilde{B}}\varphi_1 M_0[\varphi_1]dV}$$

are modal neutron generation times, $\rho_1^s = 1 - (1/k_1)$ is the subcriticality of the regional mode, and k_1 is the eigenvalue of the first (regional) lambda mode: $L_0[\varphi_1] = (1/k_1)M_0[\varphi_1]$. For the fundamental (global) lambda mode we assume criticality ($k_0 = 1$, so we have $L_0[\varphi_0] = M_0[\varphi_0]$).

Direct reactivities $\delta\rho_{00}$, $\delta\rho_{11}$ and cross reactivities $\delta\rho_{01}$, $\delta\rho_{10}$ are given by the formulae:

$$\delta\rho_{mn} = \frac{\int_{\widetilde{B}}\varphi_m\delta H[\varphi_n]dV}{\int_{\widetilde{B}}\varphi_m M_0[\varphi_m]dV} \quad m,n = 0,1$$

In principle, reactivities can be written as the sum

$$\delta\rho_{mn} = \delta\rho^F_{mn} + \delta\rho^C_{mn} + \delta\rho_{\mathrm{ext},m,n}$$

of three contributions. One contribution due to internal feedbacks loops due to temperature and void effects, $\delta\rho^F_{mn}$. A second contribution $\delta\rho^C_{mn}$ arises from feedback loops due to the automatic control system. And the third contribution $\delta\rho_{\mathrm{ext},mn}$ is due to external effects that can be described as additive terms appearing in the perturbation δL of the leak operator.

To continue, we use symmetry consideration inspired by Turso et al. (1997). Let us suppose that the extrapolated core is composed by two extrapolated half-core regions, a and b : $\widetilde{B} = \widetilde{B}_a \cup \widetilde{B}_b$.

Furthermore, we assume that there is a *one-to-one* and *onto* **symmetry transformation** $\vec{r}' = \mathrm{T}(\vec{r})$ that maps $\widetilde{B}_a$ onto $\widetilde{B}_b$, and $\widetilde{B}_b$ onto $\widetilde{B}_a$, being: $\mathrm{T}(\vec{r}') = \mathrm{T}(\mathrm{T}(\vec{r})) = \mathrm{T}^2(\vec{r}) = \vec{r}$.

The fundamental mode is symmetric under T : $\varphi_0(\mathrm{T}(\vec{r})) = \varphi_0(\vec{r})$ for every $\vec{r} \in \widetilde{B}$.

The first mode is anti-symmetrical under T : $\varphi_1(\mathrm{T}(\vec{r})) = -\varphi_1(\vec{r})$ for every $\vec{r} \in \widetilde{B}$.

Now, $M_o[\cdot]$ is a linear scalar operator that has a spatial dependence: $M_o[\varphi] = \nu\Sigma_{f,0}(\vec{r})\varphi(t,\vec{r})$. We suppose that $\Sigma_{f,0}(\mathrm{T}(\vec{r})) = \Sigma_{f,0}(\vec{r})$ for every $\vec{r} \in \widetilde{B}$ (as consequence of a symmetrical fuel distribution). Then:

$$\int_{\widetilde{B}_a}\varphi_m M_0[\varphi_m]dV = \int_{\widetilde{B}_b}\varphi_m M_0[\varphi_m]dV = \frac{1}{2}\int_{\widetilde{B}}\varphi_m M_0[\varphi_m]dV$$

Taking these last equalities into account, it is possible to split the feedback reactivities as the sum of two terms, one corresponding to region a and the other to region b:

$$\delta\rho_{mn} = \frac{1}{2}\left(\frac{\int_{\widetilde{B}_a}\varphi_m\delta H[\varphi_n]dV}{\int_{\widetilde{B}_a}\varphi_m M_0[\varphi_m]dV} + \frac{\int_{\widetilde{B}_b}\varphi_m\delta H[\varphi_n]dV}{\int_{\widetilde{B}_b}\varphi_m M_0[\varphi_m]dV}\right) = \frac{1}{2}(\delta\rho^a_{mn} + \delta\rho^b_{mn}) \quad m,n = 0,1$$

Now, let us introduce the symmetric $\delta H_s = \delta M_s - \delta L_s$ and the anti-symmetric $\delta H_a = \delta M_a - \delta L_a$ operators defined by:

$$\delta H_s = \frac{1}{2}(\delta H(\vec{r}) + \delta H(\mathrm{T}(\vec{r})))$$

$$\delta H_a = \frac{1}{2}(\delta H(\vec{r}) - \delta H(\mathrm{T}(\vec{r})))$$

Due to the different symmetry of φ_0 and φ_1, it follows that:

$$\delta\rho_{ii} = \frac{\int_{\widetilde{B}}\varphi_i\delta H_s[\varphi_i]dV}{\int_{\widetilde{B}}\varphi_i M_0[\varphi_i]dV}, \quad i = 0,1 \tag{7}$$

$$\delta\rho_{ij} = \frac{\int_{\widetilde{B}}\varphi_i\delta H_a\left[\varphi_j\right]dV}{\int_{\widetilde{B}}\varphi_i M_0[\varphi_i]dV}, \quad i \neq j,\ i,j = 0,1 \tag{8}$$

The direct mode reactivities $\delta\rho_{00}$ and $\delta\rho_{11}$ involve the **symmetric operator** δH_s, while the cross mode reactivities $\delta\rho_{01}$ and $\delta\rho_{10}$ involve the **anti-symmetric operator** δH_a. It follows that in principle they could be all different.

Now, we need formulae for the direct ($\delta\rho_{00}$, $\delta\rho_{11}$) and cross reactivities ($\delta\rho_{01}, \delta\rho_{10}$) as functions of thermal–hydraulics core variables and control system variables. From (7) we see that the direct reactivities, due to the appearance of δH_s in both formulae, must show a similar dependence of the thermo-hydraulic and control variables, although they are not proportional one to the other.

However, from (8) taking into account that $\langle\varphi_0, \delta H_a[\varphi_1]\rangle = \langle\varphi_1, \delta H_a[\varphi_0]\rangle$, we see that **the cross reactivities are proportional**

$$\delta\rho_{0,1} = f_{01}\delta\rho_{1,0} \tag{9}$$

Here, by definition f_{01} is a proportionality factor:

$$f_{01} = \frac{\int_{\widetilde{B}}\varphi_1 M_0[\varphi_1]dV}{\int_{\widetilde{B}}\varphi_0 M_0[\varphi_0]dV}$$

The proportionality factor depends of the spatial distribution of the macroscopic fission cross sections of the fuel in the core and of the lambda modes.

Due to the usual symmetric distribution of fuel in reactor core, and to the differences in the spatial fields of the fundamental and the first lambda modes, we can expect that the non-dimensional factor f_{01} be almost an order of magnitude less than 1.

So the strength of the coupling between the modes is different. The regional mode couples weakly with the global mode, in comparison with the strength of the coupling of the global mode with the regional one. This must be taken into due account in the analysis of the modal-nodal model.

Remark 1. If, instead of beginning with the partial integral–differential Eq. (3), we begin from the system (1) and (2), and take into account (4) and the approximation to the field of delayed neutron emitters $c(t,\vec{r}) \approx C_0(t)\varphi_0(\vec{r}) + C_1(t)\varphi_1(\vec{r})$, and we follow the above mentioned method of derivation, we obtain the equivalent system of four coupled ordinary differential equations, more convenient for digital simulation purposes:

$$\Lambda_0 \frac{dP_0(t)}{dt} = (\delta\rho_{00} - \beta)(P_* + P_0(t)) + \delta\rho_{01}P_1(t) + \Lambda_0\lambda C_0 \tag{10}$$

$$\frac{d}{dt}C_0 = \frac{\beta}{\Lambda_0}(P_* + P_0(t)) - \lambda C_0 \tag{11}$$

$$\Lambda_1 \frac{dP_1(t)}{dt} = \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s - \beta + \delta\rho_{11})P_1(t) + \Lambda_1\lambda C_1 \tag{12}$$

$$\frac{d}{dt}C_1 = \frac{\beta}{\Lambda_1}P_1(t) - \lambda C_1 \tag{13}$$

2.2. Thermo-hydraulic nodal model

As said before, the core B is split into two symmetric halves (nodes) B_a and B_b. A fuel temperature and a channel void fraction are assigned as state variables to each half core.

2.2.1. Fuel to coolant heat transfer

Introducing: C (fuel heat capacity), h (a fuel to coolant heat transfer coefficient), T_* (a steady state temperature), P_* (a steady state power), T_a, P_a (temperature and power assigned to B_a) and T_b, P_b (temperature and power assigned to B_b) we postulate these linear heat transfer equations:

$$\frac{C}{2}\frac{d(T_a - T_*)}{dt} = \left(P_a - \frac{P_*}{2}\right) - \frac{h}{2}(T_a - T_*) \tag{14}$$

$$\frac{C}{2}\frac{d(T_b - T_*)}{dt} = \left(P_b - \frac{P_*}{2}\right) - \frac{h}{2}(T_b - T_*) \tag{15}$$

If $T_0 = \frac{1}{2}(T_a + T_b)$ is the **average temperature** and $P = P_a + P_b$ is the **instantaneous power** of the reactor, from (14) and (15) it follows, if by definition $\delta T_0 = T_0 - T_*$:

$$C\frac{d\delta T_0}{dt} = (P - P_*) - h\delta T_0 \tag{16}$$

If $\Delta T = (T_b - T_a)$ and $P_1 = P_b - P_a$ then, from (14) and (15) we obtain:

$$C\frac{d(\Delta T)}{dt} = 2P_1 - h\Delta T \tag{17}$$

Introducing the operator:

$$\widehat{L}_T[f(t)] = \frac{df(t)}{dt} + \frac{h}{C}f(t)$$

Eqs. (16) and (17) may be recast as follows:

$$\widehat{L}_T[\delta T_0(t)] = \frac{1}{C}(P - P_*) = \frac{1}{C}P_0 \tag{18}$$

$$\widehat{L}_T[\Delta T(t)] = \frac{2}{C}P_1 \tag{19}$$

Here, by definition, $P_0 = P - P_*$ is **the perturbation** of the total core power relative to its steady state. $P_1 = P_b - P_a$, may be considered as a **perturbation in the reactor's symmetry** as expressed by the difference in the instantaneous power between the two halves of the core.

If we suppose that $P_0(t)$ and $P_1(t)$ are known from $-\infty$ to the present instant, solving Eqs. (18) and (19) we obtain

$$\delta T_0(t) = \frac{1}{h}\int_{-\infty}^{t} h_T(t-u)P_0(u)du = G_T\int_0^{\infty} h_T(u)P_0(t-u)du \tag{20}$$

$$\Delta T(t) = \frac{2}{h}\int_{-\infty}^{t} h_T(t-u)P_1(u)du = 2G_T\int_0^{\infty} h_T(u)P_1(t-u)du \tag{21}$$

with $h_T(t) = \frac{h}{C}e^{-\frac{h}{C}t}H(t)$, ($H(t)$ is Heaviside's unit step function at the origin), $G_T = \frac{1}{h}$ is a static thermal gain factor introduced in order to have $\int_0^{\infty} h_T(t)dt = 1$. Taking into account that $h_T(t)$ is positive in $(0,+\infty)$, an **average temperature delay time** t_T can be introduced by the formula:

$$t_T = \int_0^{\infty} t h_T(t)dt = \frac{C}{h}$$

Under quasi-static conditions $\delta T_0(t) \approx G_T P_0(t)$ and $\Delta T_0(t) \approx 2G_T P_1(t)$

Remark 2. The thermal static gain and the normalized impulse response function could be reinterpreted, allowing a generalization of the simple nodal heat transfer model considered in this work.

2.2.2. Steam void dynamics

Let us consider the void fractions α_a in B_a and α_b in B_b. The mean void fraction in the core is

$$\alpha \approx \frac{1}{2}(\alpha_a + \alpha_b)$$

and the deviation between half core void fractions is

$$\Delta\alpha \approx (\alpha_b - \alpha_a)$$

If α_* is the void fraction in the symmetric steady state with reactor power P_*, we define the perturbations

$$\delta\alpha_a = \alpha_a - \alpha_* \quad \delta\alpha_b = \alpha_b - \alpha_* \quad \delta\alpha = \frac{1}{2}(\delta\alpha_a + \delta\alpha_b)$$

Following the classical March-Leuba's ROM, the connection between $\delta\alpha$ and δT_0 is frequently assumed to be of the type:

$$\frac{d^2\delta\alpha}{dt^2} + b_1\frac{d\delta\alpha}{dt} + b_2\delta\alpha = b_3\delta T_0 + b_4\frac{d\delta T_0}{dt} \tag{22}$$

The derivation of Eq. (22) as well as a set of analytical formulae for b_1, b_2, b_3 and b_4, in the case of the direct loop mechanism, is done substituting an exponential $e^{-s\tau}$ in Laplace's variable by its Padé's $(2,2)$ approximation. Now (as shown, for example, in the Appendix of Henning (1999)), it is possible to derive the following approximate formulae for the parameters of (22)

$$b_1 \approx \frac{6}{\tau} \tag{23}$$

$$b_2 \approx \frac{12}{\tau^2} \tag{24}$$

$$b_3 \approx -\frac{6\Theta H^2}{\tau^2} \tag{25}$$

$$b_4 \approx -\frac{\Theta H^2}{\tau} \tag{26}$$

Here $\tau \approx \frac{H}{\bar{V}_0}$ is an estimation of the residence time of steam bubbles in the channel, H is channel height, Θ is a positive parameter (that in our case has a different meaning that in Henning (1999)), and $\bar{V}_0$ is a mean void propagation velocity. Then, from formulae Eqs. (23) and (24) it follows that the natural frequency of the void oscillator is

$\omega_0 = \sqrt{b_2} = \frac{\sqrt{12}}{\tau}$ and the friction factor $\zeta_0 = \frac{b_1}{2\omega_0} = \frac{3}{\sqrt{12}} < 1$. However, if a higher order Padé's approximation is used, or if a different mathematical model of void feedback is constructed, other equations relating excess void reactivity with excess fuel temperature are obtained, instead of Eq. (22).

So let us generalize Eq. (22) for excess void reactivity, and from now on, let us work with a **generic impulse response function** $h_V(t)$ that induces a convolution operator

$$\mathcal{H}_V(q(t)) = \int_0^\infty h(u)q(t-u). \tag{27}$$

Then we write:

$$\delta\alpha_a(t) = G_{V,a}\mathcal{H}_{V,a}(\delta T_a(t))$$
$$\delta\alpha_b(t) = G_{V,b}\mathcal{H}_{V,b}(\delta T_b(t))$$

Here $\delta T_a(t) = T_a(t) - T_*$ and $\delta T_b(t) = T_b(t) - T_*$ so $\delta T_0(t) = \frac{1}{2}(\delta T_a(t) + \delta T_b(t))$.

The static gain factors $G_{V,a},\ G_{V,b}$ are introduced in order to have

$$\int_0^\infty h_{V,a}(u)du = \int_0^\infty h_{V,b}(u)du = 1.$$

If the two halves are symmetric enough: $h_{V,a}(u) = h_{V,b}(u) = h_V(u)$ and $G_{V,a} = G_{V,b} = G_V$, then:

$$\delta\alpha(t) = G_V\mathcal{H}_V(\delta T_0(t)) \tag{28}$$

$$\Delta\alpha(t) = G_V\mathcal{H}_V(\Delta T(t)) \tag{29}$$

The oscillatory impulse response function $h_V(t)$ is not suitable to define a useful mean void delay time t_V by means of the integral $\int_0^\infty th_V(t)dt$ because, as a consequence of the changes of sign in $h_V(t)$, this integral underestimates the delay.

However, as shown elsewhere (Suárez-Ántola, 2007), a suitable average time lag may be estimated using a damped exponential $E(t)$ that gives the decreasing amplitudes of the oscillations in the impulse response function $h_V(t) : E(t) = \mu e^{-\mu t}$. The positive normalizing factor μ is introduced in order to have $\int_0^\infty E(t)dt = 1$

Then an **average void delay time** can be estimated by the formula:

$$t_V \approx \theta\,\bar{t} = \theta\int_0^\infty tE(t)dt = \theta\frac{1}{\mu}$$

The non-dimensional factor θ can be taken as 1. When the relation between void fraction and fuel temperature is the classical one (Eq. (22)), $\mu = \frac{b_1}{2}$ and $t_V \approx \theta\frac{2}{b_1} \approx \frac{2}{b_1}$: the **time lag increases with decreasing damping** in void fraction oscillations.

Under quasi-static conditions: $\delta\alpha(t) \approx G_V\delta T_0(t),\ \Delta\alpha(t) \approx G_V\Delta T(t)$

2.3. A simplified nodal model of the control system

We consider again the core split into two symmetric halves (nodes) B_a and B_b. Control feedback reactivities are assigned to each half core. Here we assume that the desired control goal is to maintain a constant power $P_*/2$ in each half core. The error signals for this purpose are

$$P_a(t-u) - \frac{P*}{2}, \quad P_b(t-u) - \frac{P*}{2}$$

Under this assumption, it is not necessary to consider the control requirements of load following, involving the modification of recirculation flow by the control system. Only the lumped effect of the movement of control rods on the reactivity of each half core must be modeled. The distributed actuator outputs in the nodal model are lumped in two rod insertion variables,

$$y_a = \frac{y_*}{2} + \delta y_a, \quad y_b = \frac{y_*}{2} + \delta y_b$$

for the half-core B_a and B_b, respectively. When the reactor is in the desired steady state, $y_a = y_b = \frac{y_*}{2}$. We assume a linear control. The impulse response functions of the control loops that relate the error signals with the variations δy_a and δy_b in the reactivity control variables are:

$$G_{a,c}h_{a,c}(t), \quad G_{b,c}h_{b,c}(t)$$

with $\int_0^\infty h_{a,c}(t)dt = 1$ and $\int_0^\infty h_{v,c}(t)dt = 1$ Then we have:

$$\delta y_a = G_{a,c}\int_0^\infty h_{a,c}\left(P_a(t-u) - \frac{P_*}{2}\right)du \tag{30}$$

$$\delta y_b = G_{b,c}\int_0^\infty h_{b,c}\left(P_b(t-u) - \frac{P_*}{2}\right)du \tag{31}$$

The hypothesis of symmetry of the core, **extended to the control system**, allows us to put $h_{a,c}(t) = h_{b,c}(t) = h_c(t), G_{a,c} = G_{b,c} = G_c$. From the relations

$$P_a(t) + P_b(t) = P(t) - P_* = P_0(t)$$
$$P_a(t) - P_b(t) = P_1(t)$$

and Eqs. (30) and (31), the sum and the difference of the actuators outputs are, respectively:

$$\delta y_0 = \delta y_a + \delta y_b = G_c\mathcal{H}_c(P_0(t))du \tag{32}$$

$$\Delta y = \delta y_a - \delta y_b = G_c\mathcal{H}_c(P_1(t))du \tag{33}$$

with

$$\mathcal{H}_c(q(t)) = \int_0^\infty h_c(u)q(t-u)du \tag{34}$$

Now the error signals $P_0(t)$ and $P_1(t)$ corresponds respectively to the global and the regional modes.

2.4. Combination of the two-mode neutron kinetic model with the two-node thermo-hydraulic model and automatic control system model through feedback reactivities

To connect the modal model of the neutronics with the nodal models for the thermo-hydraulics and the automatic control, let as introduce a temperature field, a void fraction field and a rod insertion field:

$$T(t,\vec{r}) = T_*\varphi_0(\vec{r}) + \delta T_0(t)\varphi_0(\vec{r}) + \Delta T(t)\varphi_1(\vec{r}) \tag{35}$$

$$\alpha(t,\vec{r}) = \alpha_*\varphi_0(\vec{r}) + \delta\alpha_0(t)\varphi_0(\vec{r}) + \Delta\alpha(t)\varphi_1(\vec{r}) \tag{36}$$

$$y(t,\vec{r}) = y_*\varphi_0(\vec{r}) + \delta y_0(t)\varphi_0(\vec{r}) + \Delta y(t)\varphi_1(\vec{r}) \tag{37}$$

Here $T_*\varphi_0(\vec{r})$, $\alpha_*\varphi_0(\vec{r})$ and $y_*\varphi_0(\vec{r})$ are certain temperature, void fraction and rod insertion profiles given in terms of the temperature T_*, void fraction α_* and global insertion y_* assigned to the steady state. Substituting Eqs. (35)–(37) in the evolution operator:

$$\begin{aligned}H(t,T(t,\vec{r}),\alpha(t,\vec{r}),y(t,\vec{r})) &= H_0(t,T_*\varphi_0(\vec{r}),\alpha_*\varphi_0(\vec{r}),y_*\varphi_0(\vec{r})) + \delta H(t,\delta T_0(t)\varphi_0(\vec{r}) \\ &+\Delta T(t)\varphi_1(\vec{r}),\delta\alpha_0(t)\varphi_0(\vec{r}) + \Delta\alpha(t)\varphi_1(\vec{r}) + \delta y_0(t)\varphi_0(\vec{r}) \\ &+\Delta y(t)\varphi_1(\vec{r}))\end{aligned}$$

Developing in terms of the perturbations in the temperature, the void, and the rod insertion fields up to the linear terms, we find:

$$\delta H = \delta H_s + \delta H_a$$

$$\delta H_s[\cdot] = \delta T_0\widehat{S}_T[\cdot] + \delta\alpha_0\widehat{S}_V[\cdot] + \delta y_0\widehat{S}_C[\cdot] \tag{38}$$

$$\delta H_a[\cdot] = \delta T_0\widehat{A}_T[\cdot] + \delta\alpha_0\widehat{A}_V[\cdot] + \delta y_0\widehat{A}_C[\cdot] \tag{39}$$

In these relation $\widehat{S}$ and $\widehat{A}$ are symmetric and antisymmetric linear operators, respectively, corresponding to the effects on the production and loss operators due to temperature, void fraction and control variables variations.

Given any operator $\widehat{O}$, we introduce the inner product notation:

$$\left\langle \varphi, \widehat{O}[\psi] \right\rangle = \int_B \phi \widehat{O}[\psi] dV$$

Then from Eqs. (7) and (8) and (38) and (39) we obtain the following formulae for the direct and cross feedback reactivities.

Direct, global mode:

$$\delta\rho^F_{00} = \frac{\langle \varphi_0, \delta H_s[\varphi_0]\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle} = c^{FT}_{00}\delta T_0 + c^{FV}_{00}\delta\alpha + c^{FC}_{00}\delta y_0$$

The direct Doppler c^{FT}_{00}, void c^{FV}_{00} and control c^{FC}_{00} feedback coefficients for the global mode are given by:

$$c^{FT}_{00} = \frac{\left\langle \varphi_0, \widehat{S}_T[\varphi_0]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle}, \quad c^{FV}_{00} = \frac{\left\langle \varphi_0, \widehat{S}_V[\varphi_0]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle},$$
$$c^{FC}_{00} = \frac{\left\langle \varphi_0, \widehat{S}_C[\varphi_0]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle},$$

Both c^{FT}_{00} and c^{FV}_{00} are negative. As reactivity decreases with rod insertion, c^{FC}_{00} must be negative also.

Direct, regional mode:

$$\delta\rho^F_{11} = \frac{\langle \phi_1, \delta H_s[\varphi_1]\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle} = c^{FT}_{11}\delta T_0 + c^{FV}_{11}\delta\alpha + c^{FC}_{11}\delta y_0$$

The direct Doppler c^{FT}_{11}, void c^{FV}_{11} and control c^{FC}_{11} feedback coefficients for the regional mode are given by:

$$c^{FT}_{11} = \frac{\left\langle \varphi_1, \widehat{S}_T[\varphi_1]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle}, \quad c^{FV}_{11} = \frac{\left\langle \varphi_1, \widehat{S}_V[\varphi_1]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle},$$
$$c^{FC}_{11} = \frac{\left\langle \varphi_1, \widehat{S}_C[\varphi_1]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle},$$

Cross, from regional to global:

$$\delta\rho^F_{01} = \frac{\langle \varphi_0, \delta H_a[\varphi_1]\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle} = c^{FT}_{01}\Delta T + c^{FV}_{01}\Delta\alpha + c^{FC}_{01}\Delta y$$

The cross Doppler c^{FT}_{01}, void c^{FV}_{01} and control c^{FC}_{01} feedback coefficients are given by:

$$c^{FT}_{01} = \frac{\left\langle \varphi_0, \widehat{A}_T[\varphi_1]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle}, \quad c^{FV}_{01} = \frac{\left\langle \varphi_0, \widehat{A}_V[\varphi_1]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle},$$
$$c^{FC}_{01} = \frac{\left\langle \varphi_0, \widehat{A}_C[\varphi_1]\right\rangle}{\langle \varphi_0, M_0[\varphi_0]\rangle},$$

Cross, from global to regional:

$$\delta\rho^F_{10} = \frac{\langle \varphi_1, \delta H_a[\varphi_0]\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle} = c^{FT}_{10}\Delta T + c^{FV}_{10}\Delta\alpha + c^{FC}_{10}\Delta y$$

The cross Doppler c^{FT}_{10}, void c^{FV}_{10} and control c^{FC}_{10} feedback coefficients are given by:

$$c^{FT}_{10} = \frac{\left\langle \varphi_1, \widehat{A}_T[\varphi_0]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle}, \quad c^{FV}_{10} = \frac{\left\langle \varphi_1, \widehat{A}_V[\varphi_0]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle}$$
$$c^{FC}_{10} = \frac{\left\langle \varphi_1, \widehat{A}_C[\varphi_0]\right\rangle}{\langle \varphi_1, M_0[\varphi_1]\rangle}$$

As consequence of the relation of proportionality (9) between cross feedback reactivities, we obtain for the cross Doppler, void and control feedback coefficients:

$$c^{FT}_{01} = f_{01}c^{FT}_{10}, \quad c^{FV}_{01} = f_{01}c^{FV}_{10}, \quad c^{FC}_{01} = f_{01}c^{FC}_{10}$$

Now, let us split direct and cross reactivities in the sum of a thermal–hydraulic feedback term $\delta\rho^F_{ij}$ and a control term $\delta\rho^C_{ij}$:

$$\delta\rho_{ij} = \delta\rho^F_{ij} + \delta\rho^c_{ij} \quad i,j = 0,1,\ldots$$

The temperatures are related with the fundamental and first mode amplitudes by Eqs. (16) and (17). The void fractions $\delta\alpha$ and $\Delta\alpha$ are related with the temperatures $\delta T_0 = T_0(t) - T_*$ and ΔT by the Eqs. (28) and (29). Control variables are related with the fundamental and first mode amplitude by Eqs. (32) and (33). So, we obtain the following formulae for direct and cross feedback reactivities in terms of temperature and power perturbations:

$$\begin{aligned}
\delta\rho^F_{00} &\approx c^{FT}_{00}\delta T_0(t) + c^{FV}_{00}G_V\mathcal{H}_V(\delta T_0(t))\\
\delta\rho^C_{00} &\approx c^{FC}_{00}G_C\mathcal{H}_c(P_0(t))\\
\delta\rho^F_{11} &\approx c^{FT}_{11}\delta T_0(t) + c^{FV}_{11}G_V\mathcal{H}_V(\delta T_0(t))\\
\delta\rho^C_{11} &\approx c^{FC}_{11}G_C\mathcal{H}_c(P_0(t))\\
\delta\rho^F_{10} &\approx c^{FT}_{10}\Delta T + c^{FV}_{10}G_V\mathcal{H}_V(\Delta T(t))\\
\delta\rho^C_{10} &\approx c^{FC}_{10}G_C\mathcal{H}_c(P_0(t))
\end{aligned} \tag{40}$$

with $\mathcal{H}_V(\cdot)$ defined in (27) and $\mathcal{H}_c(\cdot)$ defined in (34).

Introducing the impulse response function $h_{TV}(u) = \int_0^\infty h_T(w)h_V(u-w)dw$, from these equations for the feedback reactivities in terms of temperatures and from Eqs. (20), (21), (28) and (29) we obtain:

$$\begin{aligned}
\delta\rho^F_{00} &\approx c^{FT}_{00}G_T\mathcal{H}_T(P_0(t)) + c^{FV}_{00}G_TG_V\mathcal{H}_{TV}(P_0(t)) + c^{FC}_{00}G_c\mathcal{H}_c(P_0(t))\\
\delta\rho^F_{11} &\approx c^{FT}_{11}G_T\mathcal{H}_T(P_0(t)) + c^{FV}_{11}G_TG_V\mathcal{H}_{TV}(P_0(t)) + c^{FC}_{00}G_c\mathcal{H}_c(P_0(t))\\
\delta\rho^F_{10} &\approx c^{FT}_{10}2G_T\mathcal{H}_T(P_1(t)) + c^{FV}_{10}2G_TG_V\mathcal{H}_{TV}(P_1(t)) + c^{FC}_1G_c\mathcal{H}_c(P_1(t))
\end{aligned} \tag{41}$$

with

$$\mathcal{H}_T(q(t)) = \int_0^\infty h_T(u)q(t-u)du$$
$$\mathcal{H}_{TV}(q(t)) = \int_0^\infty h_{TV}(u)q(t-u)du$$

2.5. Auxiliary formulae for the study of nonlinear oscillations

To study stability and bifurcation with analytical technics it is convenient to obtain first the oscillator equations for the global and regional mode amplitudes.

In order to derive a coupled nonlinear oscillator equations in terms of the fundamental $P_0(t)$ and first $P_1(t)$ modes, the operator $L_T[\cdot]$ will be applied to Eqs. (5) and (6), or to a simplified version of these equations as the one given in Section 3.

To obtain the aforementioned oscillator equations it will be necessary to calculate $\widehat{L}_T\left[\delta\rho^F_{ij}\right]$, $\widehat{L}_C\left[\delta\rho^C_{ij}\right]$ and $\frac{d}{dt}\widehat{L}_T\left[\delta\rho^F_{ij}\right]$.

Taking into account the Eq. (40) that give the feedback reactivities as linear functions of the perturbations in fuel temperature and void fraction, and Eqs. (16) and (17) for heat transfer, the following expressions are obtained obtained (the dot over a function of time represents, as usual, the time derivative of the function):

$$
\begin{aligned}
&\widehat{L}_T[\delta\rho^F_{00}] = c^{FT}_{00}\frac{P_0(t)}{C} + c^{FV}_{00}\frac{G_V}{C}\mathcal{H}_V\left(\dot{P}_0(t)\right)\\
&\widehat{L}_T[\delta\rho^C_{00}] = c^{FC}_{00}G_c\mathcal{H}_c\left(\dot{P}_0(t) + \frac{h}{C}P_0(t)\right)\\
&\frac{d}{dt}\widehat{L}_T[\delta\rho^F_{00}] = c^{FT}_{00}\frac{\dot{P}_0(t)}{C} + c^{FV}_{00}\frac{G_V}{C}\mathcal{H}_V\left(\dot{P}_0(t)\right)\\
&\widehat{L}_T[\delta\rho^F_{11}] \approx c^{FT}_{11}\frac{P_0(t)}{C} + c^{FV}_{11}\frac{G_V}{C}\mathcal{H}_V(P_0(t))\\
&\widehat{L}_T[\delta\rho^C_{11}] = c^{FC}_{11}G_c\mathcal{H}_c\left(\dot{P}_0(t) + \frac{h}{C}P_0(t)\right)\\
&\frac{d}{dt}\widehat{L}_T[\delta\rho^F_{11}] \approx c^{FT}_{11}\frac{\dot{P}_0(t)}{C} + c^{FV}_{11}\frac{G_V}{C}\mathcal{H}_V\left(\dot{P}_0(t)\right)\\
&\widehat{L}_T[\delta\rho^F_{10}] \approx 2c^{FT}_{10}\frac{P_1(t)}{C} + 2c^{FV}_{10}\frac{G_V}{C}\mathcal{H}_V(P_1(t))\\
&\widehat{L}_T[\delta\rho^C_{10}] = c^{FC}_{10}G_c\mathcal{H}_c\left(\dot{P}_1(t) + \frac{h}{C}P_1(t)\right)\\
&\frac{d}{dt}\widehat{L}_T[\delta\rho^F_{10}] \approx 2c^{FT}_{10}\frac{\dot{P}_1(t)}{C} + 2c^{FV}_{10}\frac{G_V}{C}\mathcal{H}_V\left(\dot{P}_1(t)\right)
\end{aligned}
\tag{42}
$$

with $\mathcal{H}_V(\cdot)$ given by (27) and $\mathcal{H}_c(\cdot)$ defined in (34).

Remark 3. The first objective of the present work has been obtained with Eqs. (5) and (6) or the equivalent set (10)–(13) jointly with Eqs. (40), (16), (17), (28), (29), (32) and (33).

3. Prompt-jump approximation

The prompt-jump approximation is done for **only one group of delayed neutrons** emitters so it can be applied in this case.

From Eqs. (5) and (6), for the fundamental and the first mode, applying the **prompt jump approximation** we obtain:

$$
\begin{aligned}
&\delta\rho_{00}(P_* + P_0(t)) + \delta\rho_{01}P_1(t)\\
&\quad + \beta\int_0^\infty D(u)(P_0(t-u) - P_0(t))du \approx 0
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&(\rho^s_1 + \delta\rho_{11})P_1(t) + \delta\rho_{10}(P_* + P_0(t))\\
&\quad + \beta\int_0^\infty D(u)(P_1(t-u) - P_1(t))du \approx 0
\end{aligned}
\tag{44}
$$

Let us consider the differential operator of the delayed neutrons: $L_d[\cdot] = \frac{d}{dt}[\cdot] + \lambda[\cdot]$. Applying it to the convolution of $D(t) = \lambda e^{-\lambda t}$ with any well behaved function $f(t)$ we obtain $L_d\int_0^\infty D(u)f(t-u)du = \lambda f(t)$.

Taking this last equality into account and applying the operator L_d to both members of Eqs. (43) and (44), neglecting $\delta\rho_{0,1}P_1$ relative to $(\beta_0 - \delta\rho_{00})P_0$ in the first and $\delta\rho_{1,0}P_0$ relative to $(\beta_1 + |\rho^s_1| - \delta\rho_{11})P_1$, it follows:

$$
\begin{aligned}
&(\beta - \delta\rho_{00})\frac{dP_0}{dt} - \left(\frac{d\delta\rho^F_{00}}{dt} + \lambda\delta\rho_{00}\right)P_0\\
&\quad = \left(\frac{d\delta\rho^F_{00}}{dt} + \lambda\delta\rho_{00}\right)P_* + \left(\frac{d\delta\rho^F_{01}}{dt} + \lambda\delta\rho_{01}\right)P_1
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&(\beta + |\rho^s_1| - \delta\rho_{11})\frac{dP_1}{dt} - \left(\frac{d\delta\rho_{11}}{dt} + \lambda(\delta\rho_{11} - |\rho^s_1|)\right)P_1\\
&\quad = \left(\frac{d\delta\rho^F_{10}}{dt} + \lambda\delta\rho_{10}\right)P_* + \left(\frac{d\delta\rho_{10}}{dt} + \lambda\delta\rho_{10}\right)P_0
\end{aligned}
\tag{46}
$$

Eqs. (45) and (46) jointly with Eq. (41) for the direct and cross reactivities are the mathematical expression of a highly simplified reduced order model of a BWR.

3.1. Equation for the linearized global mode amplitude in the prompt-jump approximation with continuous and discrete void time delays, neglecting automatic control effects

The linearized global mode equation in the prompt-jump approximation, neglecting control effects (ρ^C_{00}) is:

$$
\begin{aligned}
&\frac{dP_0}{dt} = \frac{1}{\beta}\left(\frac{d\delta\rho^F_{00}}{dt} + \lambda\delta\rho^F_{00}\right)P_*\\
&\widehat{L}_T[\delta\rho_{00}] = c^{FT}_{00}\frac{P_0(t)}{C} + c^{FV}_{00}\frac{G_V}{C}\mathcal{H}_V(P_0(t))\\
&\frac{d}{dt}\widehat{L}_T[\delta\rho_{00}] = c^{FT}_{00}\frac{\dot{P}_0(t)}{C} + c^{FV}_{00}G_V\mathcal{H}_V\left(\dot{P}_0(t)\right)\Big)
\end{aligned}
\tag{47}
$$

The thermal operator applied to the linearized equation for the global mode gives:

$$
\widehat{L}_T\left[\frac{dP_0(t)}{dt}\right] = \frac{d^2P_0(t)}{dt^2} + \frac{h}{C}\frac{dP_0(t)}{dt} = \frac{\lambda\widehat{L}_T[\delta\rho_{00}]}{\beta}P_* \frac{\frac{d}{dt}\widehat{L}_T[\delta\rho_{00}]}{\beta}P_*
$$

Taking into account Eq. (42) that give the results obtained applying the thermal operator to the direct reactivity $\delta\rho_{00}$ and its time variation, and taking into account the signs of the feedback coefficients, this equation can be rewritten thus:

$$
\frac{d^2P_0(t)}{dt^2} + a_0\frac{dP_0(t)}{dt} + b_0P_0(t) + c_0\mathcal{H}_V\left(\frac{dP_0(t)}{dt}\right) + d_0\mathcal{H}_V(P_0(t)) = 0
$$

with

$$
\begin{aligned}
&a_0 = \frac{h}{C} + \frac{|c^{FT}_{00}|}{\beta C}P_*, \quad b_0 = \frac{\lambda|c^{FT}_{00}|}{\beta C}P_*\\
&c_0 = \frac{|c^{FV}_{00}|G_V}{\beta C}P_*, \quad d_0 = \lambda\frac{|c^{FV}_{00}|G_V}{\beta C}P_*
\end{aligned}
\tag{48}
$$

Introducing a discrete time lag:

$$
\frac{d^2P_0(t)}{dt^2} + a\frac{dP_0(t)}{dt} + bP_0(t) + c\dot{P}_0(t - t_V) + dP_0(t - t_V) = 0
$$

3.2. Equation for the regional mode amplitude in the prompt-jump approximation with continuous and discrete void time delays, neglecting control effects

Neglecting control effects, the linearized equation for the regional mode amplitude is:

$$
(\beta + |\rho^s_1|)\frac{dP_1}{dt} - \lambda(-|\rho^s_1|)P_1 = \left(\frac{d\delta\rho^F_{10}}{dt} + \lambda\delta\rho_{10}\right)P_*
$$

The second member of this equality couples the regional with the global mode, so we have mode coupling in the linear approximation.

Applying the thermal operator to both members:

$$
\begin{aligned}
&(\beta + |\rho^s_1|)\left(\frac{d^2P_1}{dt^2} + \frac{h}{C}\frac{dP_1}{dt}\right) + \lambda|\rho^s_1|\left(\frac{dP_1}{dt} + \frac{h}{C}P_1\right)\\
&\quad = \left(\frac{dL_T[\delta\rho^F_{10}]}{dt} + \lambda\widehat{L}_T[\delta\rho^F_{10}]\right)P_*
\end{aligned}
$$

Taking into account Eq. (42) that give the results obtained applying the thermal operator to the cross reactivity $\delta\rho_{10}$ and its time variation, and taking into account the signs of the feedback coefficients, this equation can be rewritten thus:

$$
\frac{d^2P_1(t)}{dt^2} + a_1\frac{dP_1(t)}{dt} + b_1P_1(t) + c_1\mathcal{H}_V\left(\frac{dP_1(t)}{dt}\right) + d_1\mathcal{H}_V(P_1(t)) = 0
\tag{49}
$$

with

$$a_1 = \frac{h}{C} + \frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}$$
$$b_1 = \frac{h}{C}\frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2\lambda|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C} \quad (50)$$
$$c_1 = \frac{2|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C} \quad d_1 = \frac{2\lambda|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C}$$

Introducing a discrete void time lag:

$$\frac{d^2P_1(t)}{dt^2} + a_1\frac{dP_1(t)}{dt} + b_1P_1(t) + c_1\dot{P}_1(t-t_V) + d_1P_1(t-t_V) = 0$$

3.3. Linear stability studied with the retarded differential equations for the global and regional mode amplitudes

We see that both, for continuous and discrete time delays, the linearized equations of the global and regional mode amplitudes are of the same general type and differ only in the coefficients:

$$\frac{d^2P(t)}{dt^2} + a\frac{dP(t)}{dt} + bP(t) + c\mathcal{H}_V\left(\frac{dP(t)}{dt}\right) + d\mathcal{H}_V(P(t)) = 0 \quad (51)$$

$$\frac{d^2P(t)}{dt^2} + a\frac{dP(t)}{dt} + bP(t) + c\dot{P}(t-t_V) + dP(t-t_V) = 0 \quad (52)$$

As consequence, we will consider first the stability properties of these more general equations, and then apply the results to each one of the mode amplitudes.

The stability of the steady state $P(t) \equiv 0$ of Eq. (52) may be studied with the ansatz $P(t) = e^{zt}$ (Minorsky, 1983)

We introduce the polynomials:

$$P_1(z) = z^2 + az + b$$
$$P_2(z) = (cz + d)$$

Then, e^{zt} is a solution (in general complex valued) of the retarded equation if and only if z is a solution of the transcendental equation:

$$P_1(z) + e^{-zt_V}P_2(z) = 0 \quad (53)$$

When $t_V = 0$, this last equation reduces to the polynomial one $z^2 + (a+c)z + (b+d) = 0$. As all the coefficients are positive, the roots of this polynomial equation are negative or have negative real parts. So, as the roots are regular functions of the delay, the steady state is locally asymptotically stable if the delay is small enough.

Furthermore, for any delay **there are no positive or zero solutionszof the transcendental** Eq. (53).

So, **if the delay destabilizes the steady state, this will happen in an oscillatory manner**, for $z(t_V) = \alpha(t_V) + i\omega(t_V)$ such that **for a critical delay** $t_V = t_{V*}$ we have $z(t_{V*}) = i\omega(t_{V*})$. Substituting $z = i\omega$ in (53) we obtain:

$$\frac{P_1(i\omega)}{P_2(i\omega)} = e^{-i\omega t_V}$$

Taking the moduli:

$$\frac{|P_1(i\omega)|}{|P_2(i\omega)|} = \frac{\sqrt{(\omega^2-b)^2 + a^2\omega^2}}{\sqrt{d^2+c^2\omega^2}} = |e^{-i\omega t_V}| = 1$$

Squaring and reordering we obtain:

$$S(\mu) = |P_2(i\omega)|^2 - |P_1(i\omega)| = \mu^2 + A\mu + B = 0 \quad (54)$$

with

$$\mu = \omega^2, \quad A = a^2 - c^2 - 2b, \quad B = b^2 - d^2$$

So, the equation $z^2 + az + b + e^{-zt_V}(cz+d) = 0$ will have a purely imaginary root $z = i\omega$ if and only if $S(\mu) = 0$ has a positive root $\mu_* = \omega_*^2$. If this is the case, $\omega_* = +\sqrt{\mu_*}$ will be the critical frequency.

The corresponding **critical delay** will be given by:

$$t_{V*} = \frac{i}{\omega_*}\log_e\left(\frac{(\omega_*^2 - b) - ia\omega_*}{d + ic\omega_*}\right)$$
$$= \frac{1}{\omega_*}\arctan\left(\omega_*\frac{(ad-bc) + c\omega_*^2}{bd + (ac-d)\omega_*^2}\right)$$

Now, to identify possible bifurcation points, let us find $\frac{d}{dt_V}\alpha(t_{V*})$ following the method used in Forde and Nelson (2004). It can be shown

$$\operatorname{sign}\frac{d}{dt_V}\alpha(t_{V*}) = \operatorname{sign}\frac{dS(\mu_*)}{d\mu} \quad (55)$$

We will have a change of stability of the steady state for this critical delay if $S(\mu_*) = 0$ and $\frac{d}{d\mu}S(\mu_*) \neq 0$. The roots of $S(\mu) = \mu^2 + A\mu + B = 0$ are

$$\mu_{1,2} = \frac{1}{2}\left(-A \pm \sqrt{A^2 - 4B}\right)$$

- If $B = b^2 - d^2 < 0$ we have **a single positive root** $\mu_* = \omega_*^2 = \frac{1}{2}\left(-A + \sqrt{A^2 + 4|B|}\right)$ (the other one is negative). But $\frac{d}{d\mu}S(\mu) = 2\mu + A$ and

$$\frac{d}{d\mu}S(\mu_*) = -A + \sqrt{A^2+4|B|} + A = \sqrt{A^2+4|B|} > 0$$

From (55) it follows $\frac{d}{dt_V}\alpha(t_{V*}) > 0$: the steady state losses its stability when the delay grows greater than its critical value.

- If $B = b^2 - d^2 > 0$, then $\mu_{1,2} = \frac{1}{2}\left(-A \pm \sqrt{A^2 - 4\mid B\mid}\right)$ and in order to have positive roots it is necessary and sufficient: $A = a^2 - c^2 - 2b < 0$ $A^2 - 4\mid B\mid \geqslant 0$. If this is the case, both roots of $S(\mu) = 0$ are positive:

$$\mu_{1*} = \omega_{*1}^2 = \frac{1}{2}\left(|A| - \sqrt{A^2 - 4|B|}\right)$$
$$\mu_{2*} = \omega_{*2}^2 = \frac{1}{2}\left(|A| + \sqrt{A^2 - 4|B|}\right)$$
$$\frac{d}{d\mu}S(\mu_{1*}) = -\sqrt{A^2-4|B|} < 0$$
$$\frac{d}{d\mu}S(\mu_{2*}) = \sqrt{A^2-4|B|} > 0$$

So, when the time delay grows from zero it attains a first critical value where the steady state changes from stable to unstable. If the delay continuous growing it attains a second critical value where the steady state changes from unstable to stable.

Remark 4. Observe that $A^2 > 4|B|$ if and only if $|A| = (c^2 + 2b - a^2) > 2\sqrt{|B|} = 2\sqrt{b^2 - d^2}$ This last inequality is equivalent to:

$$c^2 + 2\left(b - \sqrt{b^2 - d^2}\right) > a^2 \quad (56)$$

So it is more restrictive that $c^2 + 2b > a^2$ (which is equivalent to $A < 0$).

3.3.1. Linear stability for the global mode amplitude

For the global mode amplitude the parameters are given in (48). According with the general results discussed above, we must calculate first $B_0 = b_0^2 - d_0^2$ to see whether it is positive or negative. We obtain:

$$B_0 = \lambda^2\left(|c_{00}^{FT}|^2 - |c_{00}^{FV}|^2 G_V^2\right)\left(\frac{P_*}{\beta C}\right)^2 \tag{57}$$

- From (57) we see that $B_0 < 0$ is equivalent to the following relation between the Doppler and void feedback coefficients and the static void gain:

  $$|c_{00}^{FT}| < |c_{00}^{FV}| G_V$$

  If this is the case, there will be a critical time delay such that when the delay is growing, and attains and overcomes this critical value, the steady state of the global mode amplitude losses its local stability. According to the mathematical model, this happens with independence of the level of steady state power P_* of the reactor.
- On the contrary, $B_0 > 0$ if the static void gain and the Doppler and void feedback coefficients are related by the opposite inequality:

  $$|c_{00}^{FT}| > |c_{00}^{FV}| G_V \tag{58}$$

  Now, there will be critical delays when and only when (56) is satisfied. For the global mode this is equivalent to the restriction:

  $$c_0^2 + 2\left(b_0 - \sqrt{b_0^2 - d_0^2}\right) > a_0^2$$

  From this last inequality and from the definition of the parameters for the global mode we derive the equivalent inequality:

  $$\left(\frac{|c_{00}^{FT}|^2 - G_V^2|c_{00}^{FV}|^2}{\beta^2 C^2}\right)P_*^2 + \frac{h^2}{C^2} + \frac{2|c_{00}^{FT}|}{\beta C}\left(\frac{h}{C} - \lambda\left(1 - \sqrt{1-\sigma^2}\right)\right)P_* < 0 \tag{59}$$

  Here, by definition:

  $$\sigma_g = \frac{G_V|c_{00}^{FV}|}{|c_{00}^{FT}|}$$

  it is always less than 1 by (58). Eq. (59) has (two) positive roots if and only if the thermal delay time is big enough:

  $$t_H = \frac{C}{h} > \frac{\left(1 + \sqrt{1-\sigma_g^2}\right)}{\left(1 - \sqrt{1-\sigma_g^2}\right)}\frac{1}{\lambda}$$

  The roots are:

  $$P_{*1,2} = \frac{q \mp \sqrt{q^2 - \left(1-\sigma_g^2\right)\frac{h^2}{C^2}}}{\left(\frac{|c_{00}^{FT}|}{\beta C}\right)\left(1-\sigma_g^2\right)}$$

  with $q = \left(\lambda\left(1 - \sqrt{1-\sigma_g^2}\right) - \frac{h}{C}\right)$. When $P_{*,1} < P_* < P_{*,2}$, the inequality (56) is verified, so the equation $S(\mu) = 0$ has two positive roots for each steady state power P_* comprised between these lower and an upper bounds. For each power there are two critical delays $t_{V*}(P_*)$ and two critical frequencies $\omega_*(t_{V*})$ such that $\alpha_*(t_{V*}) = 0$ and $\frac{d}{dt_V}\alpha_*(t_{V*}) \neq 0$. When the Doppler and void feedback coefficients and the static void gain satisfy the inequality (58), as time delay grows from zero it attains a first critical value $t_{V*,1}$ where the steady state of the global mode amplitude changes from unstable to stable ($\frac{d}{dt_V}\alpha_*(t_{V*,1}) < 0$), and then attains a second critical value $t_{V*,2}$ where this steady state changes from stable to unstable ($\frac{d}{dt_V}\alpha_*(t_{V*,2}) > 0$).

3.3.2. Linear stability for the regional mode amplitude

For the regional mode amplitude the parameters are given in (50). From these parameters we calculate:

$$B_1 = b_1^2 - d_1^2 = \left(\left(h|\rho_1^s| + 2|c_{10}^{FT}|P_*\right)^2 - \left(2|c_{10}^{FV}|G_V P_*\right)^2\right)\cdot\left(\frac{\lambda}{(\beta + |\rho_1^s|)C}\right)^2$$

- $B_1 < 0$ is possible if and only if:

  $$h|\rho_1^s| + 2|c_{10}^{FT}|P_* < 2|c_{10}^{FV}|G_V P_*$$

This can be attained only if:

$$|c_{10}^{FT}| < |c_{10}^{FV}|G_V \tag{60}$$

Now, we find a power threshold:

$$P_{*,r} = \frac{h|\rho_1^s|}{2\left(|c_{10}^{FV}|G_V - |c_{10}^{FT}|\right)} \tag{61}$$

If the steady state power is greater than the threshold, there will be a critical delay such that when crossed, the steady state of the regional mode amplitude will be destabilized.

- $B_1 > 0$ is possible if and only if:

  $$h|\rho_1^s| > 2\left(|c_{10}^{FV}|G_V - |c_{10}^{FT}|\right)P_*$$

This will be always attained when the cross Doppler and void coefficients and the static void gain satisfy the inequality:

$$|c_{10}^{FT}| > |c_{10}^{FV}|G_V$$

When the opposite inequality

$$|c_{10}^{FT}| < |c_{10}^{FV}|G_V$$

is verified, a necessary condition to have $B_1 > 0$ and not exclude the possibility of a critical delay, is a steady state power less than the threshold $P_{*,r}$ already found. When $B_1 > 0$, a sufficient condition for the existence of a critical delay is:

$$c_1^2 + 2\left(b_1 - \sqrt{b_1^2 - d_1^2}\right) > a_1^2 \tag{62}$$

Let us introduce the parameter:

$$\sigma_r(P_*) = \frac{2|c_{10}^{FV}|G_V P_*}{h|\rho_1^s| + 2|c_{10}^{FT}|P_*}$$

if $B_1 > 0$, $\sigma_r(P_*)$ is always less than 1.
From Eq. (62), if $|c_{10}^{FT}| > |c_{10}^{FV}|G_V$ this inequality is not verified by any positive steady power, so in this case there is no critical delay. If $|c_{10}^{FT}| < |c_{10}^{FV}|G_V$, this inequality could be verified when the steady power is big enough, but in this case we do not have a simple quadratic polynomial in the steady state power. This power must always be less than the threshold $P_{*,r}$ given by formula (61), in order to have $\sigma_r(P_*) < 1$.

3.4. Linear stability studied with the integral–differential the equations for the global and regional mode amplitudes using KBM method

The stability of the steady state $P(t) \equiv 0$ of Eq. (51) may be studied also with the ansatz $P(t) = e^{zt}$ (Minorsky, 1983). Nevertheless, we will approach local stability applying an asymptotic method developed mainly by Krilov, Bogoliubov, and Mitropolsky to study nonlinear oscillations (KBM method), (Jackson, 1989; Mitropolsky and Dao, 2003). Eq. (51) may be written as

$$\ddot{P}(t) = \widehat{F}\left[P, \dot{P}\right] \tag{63}$$

In our present case, $\widehat{F}\left[P, \dot{P}\right]$, is this linear integral differential operator

$$\widehat{F}\left[P, \dot{P}\right] = -bP(t) - a\dot{P}(t) + c\mathcal{H}_V\left(\dot{P}(t)\right) + d\mathcal{H}_V(P(t)) \tag{64}$$

with $\mathcal{H}_V(\cdot)$ from (27). As is summarized in Appendix A, to zero order, the KBM method begins with the ansatz $P(t) \approx \rho(t)\cos\psi(t)$ to solve Eq. (63) in a first approximation. The phase is given by $\psi(t) = \omega t + \theta(t)$, and the assumption is made that both the amplitud of oscillation $\rho(t)$ and $\theta(t) = \psi(t) - \omega t$ being **slowly varying functions of time** in comparison with the period $2\pi/\omega$. As shown in Appendix A,

$$\dot{\rho} = \frac{1}{2\pi\omega}\int_0^{2\pi} \widehat{F}\,[\rho\cos\psi, -\omega\rho\sin\psi]\sin\psi d\psi \tag{65}$$

$$\dot{\psi} = \frac{1}{2}\omega - \frac{1}{2\pi\omega\rho}\int_0^{2\pi} F\,[\rho\cos\psi, -\omega\rho\sin\psi]\cos\psi d\psi \tag{66}$$

If we substitute (64) in (65) and (66) we obtain the following equations of evolution in terms of the new dependent variables, amplitude and phase:

$$\begin{aligned}&\dot{\rho} = \alpha\rho\\ &2\alpha = -a - cH_r(\omega) + \frac{d}{\omega}H_i(\omega)\end{aligned} \tag{67}$$

$$\begin{aligned}&\dot{\psi} = \omega\\ &\omega^2 = b + dH_r(\omega) + c\omega H_i(\omega)\end{aligned} \tag{68}$$

$$\begin{aligned}H_r(\omega) &= \int_0^\infty h_V(u)\cos\omega u du\rho\\ H_i(\omega) &= \int_0^\infty h_V(u)\sin\omega u du\rho\end{aligned}$$

In the space of reactor's parameters, the critical manifold to study local stability and bifurcations is given by the equation:

$$-a - cH_r(\omega_*) + \frac{d}{\omega_*}H_i(\omega_*) = 0 \tag{69}$$

The fast frequency $\omega = \omega_*$ that appears in formula (69) is obtained solving simultaneously (67) and (68) with $\alpha = 0$.

For the global mode amplitude we have:

$$\alpha = \frac{1}{2}\left\{-\left(\frac{h}{C} + \frac{|c_{00}^{FT}|}{\beta C}P_*\right) - \left(\frac{|c_{00}^{FV}|G}{\beta C}P_*\right)H_r(\omega) + \left(\lambda\frac{|c_{00}^{FV}|G}{\beta C}P_*\right)\frac{H_i(\omega)}{\omega}\right\} \tag{70}$$

$$\omega^2 = \left(\lambda|c_{00}^{FT}|\frac{P_*}{\beta C}\right) + \left(\lambda\frac{|c_{00}^{FV}|G}{\beta C}P\right)H_r(\omega) + \left(\frac{|c_{00}^{FV}|G}{\beta C}P_*\right)\omega H_i(\omega) \tag{71}$$

For the regional mode amplitude:

$$\alpha = \frac{1}{2}\left\{-\left(\frac{h}{C} + \frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\right) - \left(\frac{2|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)H_r(\omega) + \left(\frac{2\lambda|c_{10}^{FT}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)\frac{H_i(\omega)}{\omega}\right\} \tag{72}$$

$$\omega^2 = \left(\frac{h}{C}\frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2\lambda|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\right) + \left(\frac{2\lambda|c_{10}^{FT}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)H_r(\omega) + \left(\frac{2|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)\omega H_i(\omega) \tag{73}$$

Remark 5. When α is near zero, with ω different from zero, two complex conjugate poles are dominant. The others pairs of poles are farther away to the left from the imaginary axis of the complex plane $z = \alpha + i\omega$, so they decay fast relative to the time scale of evolution of the dominant pair. As consequence, near the imaginary axis the asymptotic linear dynamics is very nearly that of a harmonic oscillator of natural frequency ω_*, damping coefficient $\zeta = \alpha/(2\omega_*)$ and decay ratio

$$\mathrm{DR} = e^{-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

From formulae (70)–(73) first the coefficient ζ, and then DR, can be calculated as function of reactor's parameters for the global and the regional mode. The variation of the decay ratio from less than one to greater than one can be predicted, for the global and the regional mode amplitudes, as function of a suitable bifurcation parameter (steady state power, void static gain or void delay time) under conditions such that a given mode stability boundary is crossed.

Remark 6. The oscillation frequency is very close to the natural frequency of the dominant oscillator due to the fact that the damping coefficient of this oscillator is sufficiently small. This can be seen as result of the convolution with the impulse response of a narrow band filter. The remaining poles correspond to oscillations with significant damping coefficients such that their contribution is not significant in determining the frequency of the oscillation.

4. Stability and bifurcation analysis of slow processes: working with the effective life time approximation

Let us begin with the coupled Eqs. (5) and (6) for the fundamental and the first mode. To derive the effective life time approximation, in the term that gives the effect of delayed neutron emitters in these equations we approximate the mode amplitudes up to linear terms in u. Neglecting higher order terms can be justified if the time scale of variation of the mode amplitudes is greater than the time scale of $D(t)$, that is $1/\lambda$, which is of the order of 10 s. Introducing the **effective life time scale** $\Lambda_e = \Lambda + \beta t_D$, being $t_D = \int_0^\infty D(u)udu = 1/\lambda$ the time constant of the delayed neutrons emitters, the equations of the modes are reduced to these ones:

$$\begin{aligned}\Lambda_e\frac{dP_0(t)}{dt} &\approx \delta\rho_{00}(P_* + P_0(t)) + \delta\rho_{01}P_1(t)\\ \Lambda_e\frac{dP_1(t)}{dt} &\approx \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11})P_1(t)\end{aligned}$$

The prompt neutron generation times, Λ_0 and Λ_1, are of the order of 10^{-4} s (or less). For ^{235}U fuel, representative values are $\beta \approx 0.0065$, $t_D = 12.8$ s, so that an estimation of the effective generation time gives $\Lambda_e = \Lambda + \beta t_D \approx \beta t_D \approx 8.3\times10^{-2}$ s, of the order of 10^{-2} s.

4.1. Uncoupled fundamental mode dynamics in the effective life time approximation

As said before, near the end of Section 2.1, the coupling due to the cross reactivity $\delta\rho_{01}$ is weak relative to that due to $\delta\rho_{10}$. In a first approximation we uncouple the fundamental mode from the regional one:

$$\Lambda_e\frac{dP_0(t)}{dt} \approx \delta\rho_{00}(P_* + P_0(t))$$

4.1.1. Derivation of an equation of damped harmonic oscillator and decay ratios for the global mode.

To study local stability it is possible to work with the linear equation:

$$\frac{dP_0(t)}{dt} \approx \frac{\delta\rho_{00}}{\Lambda_e}P_*$$

The feedback coupling is proportional to the steady power P_* of the reactor core and to the direct reactivity of the global mode.

Applying the operator

$$\widehat{L}_T[f(t)] = \frac{df(t)}{dt} + \frac{h}{C}f(t)$$

to both sides of the equation, and neglecting for the moment reactivity effects due to automatic control, from Eq. (47) for the direct feedback reactivity of the fundamental mode, and taking into account that both c_{00}^{FT} and c_{00}^{FV} were assumed to be negative, we derive:

$$\frac{d^2P_0(t)}{dt^2}+\frac{h}{C}\frac{dP_0(t)}{dt}+\omega_{gTe}^2P_0(t)+\omega_{gVe}^2\mathcal{H}_V(P_0(t))\approx 0$$

Here

$$\omega_{gTe}=\sqrt{\frac{|c_{00}^{FT}|P_*}{\Lambda_eC}}$$

$$\omega_{gVe}=\sqrt{\frac{|c_{00}^{FV}|G_VP_*}{\Lambda_eC}}$$

are certain thermal frequency and void frequency of the uncoupled global mode in the effective life time approximation, respectively, and $\mathcal{H}_V(\cdot)$ from (27).

Let us consider slow processes, with time scales small enough relative to the void time scale. Then, in $\mathcal{H}_V(P_0(t))$ the global mode amplitude $P_0(t-u)$ can be approximated up to linear terms in u, and we obtain the following equation for a damped harmonic oscillator, where t_V is the average void delay time introduced at the end of Section 2.2.2:

$$\frac{d^2P_0(t)}{dt^2}+2\zeta_{ge}\omega_{ge}\frac{dP_0(t)}{dt}+\omega_{ge}^2\cdot P_0(t)\approx 0$$

By definition:

$$\omega_{ge}=\sqrt{\frac{(|c_{00}^{FT}|+|c_{00}^{FV}|G_V)}{\Lambda_eC}P_*}=\sqrt{\omega_{gTe}^2+\omega_{gVe}^2} \tag{74}$$

$$\zeta_{ge}=\frac{1}{2\omega_{ge}}\left(\frac{h}{C}-\frac{|c_{00}^{FV}|G_Vt_VP_*}{\Lambda_eC}\right) \tag{75}$$

So, there is negative damping (instability) when ζ_{ge} is negative.

The damping ratio of the global mode

$$\mathrm{DR}_{ge}=e^{-\frac{2\pi\zeta_{ge}}{\sqrt{1-\zeta_{ge}^2}}}$$

is a function of some reactor parameters through the friction coefficient ζ_{ge}. If $\zeta_{ge}=0$ the **damping ratio** is equal to one. Introducing the static thermal gain $G_T=\frac{1}{h}$, for a given steady power P_* of the reactor, this value is attained for a critical delay time:

$$t_{Vg_*}=\frac{\Lambda_e}{G_TG_V|c_{00}^{FV}|P_*}.$$

When this happens and the delay time t_V increases again (while the steady power P_* remains constant), the friction coefficient starts to be negative, the decay ratio is greater than 1, and steady state power gets unstable. If the combination of control parameters is less than critical, then the DR is less than one and global mode amplitude tends to return to zero.

From these results, when the effective life time approximation applies, it is possible to calculate a stability boundary for the uncoupled global mode in the power flow plane.

4.1.2. Stability and bifurcations from the nonlinear integral–differential equation of the global mode in the effective lifetime approximation

Let us begin with

$$\frac{dP(t)}{dt}=\frac{\delta\rho_{00}^F}{\Lambda_e}P(t)$$

Here the power corresponding to the fundamental mode $P(t)=P_*+P_0(t)$ is the total power in the core. Let us introduce the **logarithmic power**: $x_o(t)=\ln\left(\frac{P(t)}{P_*}\right)$. Then $P=P_*e^{x_o}$ and, if we define $g(x)=e^x-1$, it follows $P-P_*=P_*g(x_0)$

The nonlinear dynamic equation for the fundamental mode uncoupled can be recast as follows:

$$\frac{dx_0(t)}{dt}=\frac{\delta\rho_{00}^F}{\Lambda_e}$$

Applying the operator $\widehat{L}_T[f(t)]=\frac{df(t)}{dt}+\frac{h}{C}f(t)$ to both members of the equation, and taking into account (47), this **nonlinear differential-integral equation for an oscillator** is finally derived:

$$\frac{d^2x_0(t)}{dt^2}+\frac{h}{C}\frac{dx_0(t)}{dt}+\omega_{gTe}^2g(x_0)+\omega_{gVe}^2\mathcal{H}_V(g(x_0(t))\approx 0$$

Here, by definition

$$\omega_{gVe}=\sqrt{\frac{|c_{00}^{FV}|G_VP_*}{\Lambda_eC}}$$

$$\omega_{gTe}=\sqrt{\frac{|c_{00}^{FT}|P_*}{\Lambda_eC}}$$

$$\omega_{ge}=\sqrt{\omega_{gTe}^2+\omega_{gVe}^2}.$$

Expanding $g(x_0(t-u))$ in powers of u up to the second order term inclusive, the convolution term can be approximated this way:

$$\mathcal{H}_V(g(x_0(t))\approx g(x_0(t))-t_Ve^{x_0}\dot{x}_0+\frac{e^{x_0}}{2}\left(\int_0^\infty h_V(u)u^2du\right)(\ddot{x}_0+\dot{x}_0^2)$$

Substituting this approximation in the nonlinear integral–differential equation, the following nonlinear oscillator differential equation is obtained:

$$\left(1+\frac{\omega_{gVe}^2}{2}\left(\int_0^\infty h_V(u)u^2du\right)\right)\frac{d^2x_0}{dt^2}+\left(\frac{h}{C}-t_V\omega_{gVe}^2+\frac{\omega_{gVe}^2}{2}\left(\int_0^\infty h_V(u)u^2du\right)\frac{dx_0}{dt}\right)\frac{dx_0}{dt}+\omega_{ge}^2g(x_0)\approx 0$$

Taking into account that

$$\frac{\omega_{gVe}^2}{2}\left(\int_0^\infty h_V(u)u^2du\right)$$

can be neglected relative to 1, the corresponding linearized equation

$$\frac{d^2x_0(t)}{dt^2}+\left(\frac{h}{C}-t_V\omega_{gVe}^2\right)\frac{dx_0(t)}{dt}+\omega_{ge}^2x_0(t)\approx 0$$

is equivalent to the equation for $P_0(t)$.

In order to be able apply Hopf's bifurcation theorem (Jackson, 1989; Nicolis, 1995), we introduce, as usual, the new variable $x_1=\frac{dx_0}{dt}$ and take the void delay time t_V as bifurcation parameter.

Then, the equation of the nonlinear oscillator can be rewritten as a nonlinear system of first order differential equations:

$$\frac{dx_0}{dt}=f_0(x_0,x_1,t_V)=x_1$$

$$\frac{dx_1}{dt}=f_1(x_0,x_1,t_V)=\left(\omega_{gVe}^2t_V-\frac{h}{C}\right)x_1-\frac{\omega_{gVe}^2}{2}\left(\int_0^\infty h_V(u)u^2du\right)x_1^2-\omega_{ge}^2g(x_0)$$

The eigenvalues of the linearized system are $\alpha(t_V)\pm i\omega(t_V)$ with $2\alpha(t_V)=\omega_{gVe}^2t_V-\frac{h}{C}$ and $\omega(t_V)=\sqrt{1-\zeta_{ge}^2}\omega_{ge}$.

The steady state solution $x_0 = 0, x_1 = 0$ is asymptotically stable when $t_V < t_{V,g_*}$ and unstable when $t_V > t_{V,g_*}$. Besides, $\alpha(t_{V,g_*}) = 0$ and $\frac{d}{dt_V}\alpha(t_V) = \omega_{gVe}^2 > 0$ for every t_V. Furthermore, when $t_V = t_{V,g_*}$, the nonlinear oscillator equation reduces to this one, with positive damping:

$$\frac{d^2x_0}{dt^2} + \frac{\omega_{gVe}^2}{2}\left(\int_0^\infty h_V(u)u^2du\right)\left(\frac{dx_0}{dt}\right)^2 + \omega_{ge}^2 g(x_0) \approx 0$$

Due to positive damping and the positivity of $g(x_0)$, the steady state solution is locally asymptotically stable when the parameter t_V is at its critical value. So according to Hopf bifurcation theorem, **a supercritical Hopf bifurcation** occurs in the nonlinear system when the parameter t_V reaches its critical value. When $t_V > t_{V,g_*}$ a stable limit cycle births, and its amplitude increases as t_V increases in an upper neighborhood of t_{V,g_*}. According to the mathematical model and remembering that the void delay time is inversely related with coolant flow through reactor's core, when the effective life time approximation applies it is possible to find conditions in the power-flow plane such that soft self-excited oscillations can be produced in the fundamental mode amplitude.

4.2. First mode dynamics in the effective lifetime approximation. Derivation of an equation of damped harmonic oscillator and decay ratios for the regional mode

Consider

$$\Lambda_e\frac{dP_1(t)}{dt} \approx \delta\rho_{10}^F(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11}^F)P_1(t) \tag{76}$$

Linearizing Eq. (76) for the **first mode amplitude** we obtain:

$$\Lambda_e\frac{dP_1(t)}{dt} \approx \delta\rho_{10}^F P_* + \rho_1^s P_1(t) \tag{77}$$

The feedback coupling term is proportional to the steady power P_* of the reactor core and to the cross reactivity of the global mode over the regional one.

Applying the operator $\widehat{L}_T[f(t)] = \frac{df(t)}{dt} + \frac{h}{C}f(t)$ to both members of (77), and neglecting external reactivity effects we obtain:

$$\frac{d^2P_1(t)}{dt^2} + \frac{h}{C}\frac{dP_1(t)}{dt} \approx -\frac{|\rho_1^s|}{\Lambda_e}\left(\frac{dP_1(t)}{dt} + \frac{h}{C}P_1(t)\right) + \frac{\widehat{L}_T[\delta\rho_{10}^F]}{\Lambda_e}P_*$$

From Eq. (42) for the cross reactivity $\delta\rho_{10}^F$ it follows:

$$\frac{d^2P_1(t)}{dt^2} + \left(\frac{h}{C} + \frac{|\rho_1^s|}{\Lambda_e}\right)\frac{dP_1(t)}{dt} + \left(\frac{|\rho_1^s|}{\Lambda_e}\frac{h}{C} + \omega_{rTe}^2\right)P_1(t) + \omega_{rVe}^2\mathcal{H}_V(P_1(t)) \approx 0$$

Here $\omega_{rTe} = \sqrt{2\frac{|c_{10}^{FT}|P_*}{\Lambda_e C}}$ is a certain thermal frequency and $\omega_{rVe} = \sqrt{2\frac{|c_{10}^{FV}|G_V P_*}{\Lambda_e C}}$ is a certain void frequency of the regional mode in the effective life time approximation.

For processes slow enough, in $\mathcal{H}_V(P_1(t))$ the regional mode amplitude can be approximated up to linear terms in u, and we obtain again an equation of a damped harmonic oscillator:

$$\frac{d^2P_1(t)}{dt^2} + 2\zeta_{re}\omega_{re}\frac{dP_1(t)}{dt} + \omega_{re}^2P_1(t) \approx 0$$

with

$$\omega_{re}^2 = \frac{h}{C}\frac{|\rho_1^s|}{\Lambda_e} + 2\frac{(|c_{10}^{FT}| + |c_{10}^{FV}|G_V)}{\Lambda_e C}P_* = \frac{h}{C}\frac{|\rho_1^s|}{\Lambda_e} + \omega_{rTe}^2 + \omega_{rVe}^2$$

$$\zeta_{re} = \frac{1}{2\omega_{re}}\left(\frac{h}{C} + \frac{|\rho_1^s|}{\Lambda_e} - 2\frac{|c_{10}^{FV}|G_V}{\Lambda_e C}t_V P_*\right).$$

So, there is negative damping (instability) when ζ_{re} is negative.

The damping ratio of the regional mode

$$\mathrm{DR}_{re} = e^{-\frac{2\pi\zeta_{re}}{\sqrt{1-\zeta_{re}^2}}}$$

is a function of some reactor parameters through its friction coefficient ζ_{re}. If $\zeta_{re} = 0$, the **damping ratio** is equal to one. Introducing the static thermal gain $G_T = \frac{1}{h}$, then for a given steady power P_*, this value is attained the critical delay time:

$$t_{V,r_*} = \frac{\Lambda_e}{2G_V|c_{10}^{FV}|P_*}\left(\frac{1}{G_T} + \frac{|\rho_1^s|C}{\Lambda_e}\right)$$

When this happens and the delay time t_V increases again (while the steady power P_* remains constant), the friction coefficient starts to take negative values, the decay ratio is greater than 1, and steady state power gets unstable. If the combination of control parameters is less than critical, then the DR is less than one and according to the linear approximation to the dynamics, the regional mode amplitude tends to return to zero.

Remark 7. From these results it is possible to calculate the stability boundary for the regional mode in the power flow plane when the effective life time approximation applies. Now we have the regional mode always coupled with the global one, even in the linear approximation. If the cross coupling disappears the regional mode is always locally stable.

4.3. Stability and bifurcations for slow processes with feedback from the control system: the global mode case

From Eqs. (5) and (6), the evolution of mode amplitudes when the effective life time approximation is done, is described by the system:

$$\Lambda_e\frac{dP_0(t)}{dt} = (\delta\rho_{00}^F + \delta\rho_{00}^C)(P_* + P_0(t)) + (\delta\rho_{01}^F + \delta\rho_{01}^C)P_1(t)$$

$$\Lambda_e\frac{dP_1(t)}{dt} = (\delta\rho_{10}^F + \delta\rho_{10}^C)(P_* + P_0(t)) + (\rho_1^s + \delta\rho_{11}^F + \delta\rho_{11}^C)P_1(t)$$

The linear approximation to the evolution of the global mode is:

$$\Lambda_e\frac{dP_0(t)}{dt} = (\delta\rho_{00}^F + \delta\rho_{00}^C)P_*$$

Applying the thermal operator to this equation we find:

$$\Lambda_e\left(\dot{P}_0(t) + \frac{h}{C}P_0(t)\right)\frac{dP_0(t)}{dt} = \left(L_T[\delta\rho_{00}^F] + L_T[\delta\rho_{00}^C]\right)P_*$$

Making the slow process approximations in $L_T[\delta\rho_{00}^F]$, as was done in Section 4.1.1, and taking into account Eq. (42) for $L_T[\delta\rho_{00}^C]$, the evolution of the fundamental mode with automatic control and in the linear approximation is given by:

$$\frac{d^2P_0(t)}{dt^2} + 2\zeta_{g,e}\omega_{g,e}\frac{dP_0(t)}{dt} + \omega_{ge}^2P_0(t) + \frac{|c_{00}^{FC}|G_c}{\Lambda_e}\mathcal{H}_c\left(\dot{P}_0(t) + \frac{h}{C}P_0(t)\right)P_* = 0$$

The damping coefficient $2\zeta_{g,e}\omega_{g,e}$ and the natural frequency ω_{ge} are given by (74) and (75), respectively, as well defined functions of neutronic and thermo-hydraulic parameters.

Introducing a representative time lag t_c assigned to the automatic control, we arrive to a retarded differential equation of the same kind already studied in Section 3.3:

$$\frac{d^2P(t)}{dt^2} + a_{0,e}\frac{dP(t)}{dt} + b_{0,e}P(t) + c_{0,c}\dot{P}(t - t_c) + d_{0,c}P(t - t_c) = 0 \tag{78}$$

Here $a_{0,e} = 2\zeta_{g,e}\omega_{g,e}$ and $b_{0,e} = \omega_{ge}^2$ are the coefficients of the underdamped harmonic oscillator that approximates the global mode dynamics in the case of slow processes, while $c_{0,c} = \frac{|c_{00}^{FC}|G_c}{\Lambda_e}P_*$ and

$d_{0,c} = \frac{|c_{00}^{FC}|G_c}{\Lambda_e}\frac{h}{C}P_*$ have their origin in the coupling of the automatic control with the fundamental mode amplitude through the control loop reactivity $\delta\rho_{00}^{C}$.

The stability of the steady state $P_0(t) \equiv 0$ of Eq. (78) may be studied with the ansatz $P(t) = e^{z_c t}$ following the method employed in Section 3.3.

We introduce the neutronic-thermo-hydraulic feedback polynomial

$$P_{TH}(z) = z^2 + a_{0,e}z + b_{0,e}$$

and the polynomial related with automatic control feedback:

$$P_c(z) = (c_{0,c}z + d_{o,c})$$

Then analogously than in Section 3.3, $e^{z_c t}$ is a solution (in general complex valued) of the retarded equation if and only if $z_c = \alpha_c + i\omega_c$ is a solution of the transcendental equation:

$$P_{TH}(z) + e^{-zt_c}P_c(z) = 0 \tag{79}$$

When the control lag $t_c = 0$, this last equation reduces to the polynomial one

$$z^2 + (a_{0,e} + c_{0,c})z + (b_{0,e} + d_{0,c}) = 0$$

As all the coefficients are positive (we suppose that $a_{0,e} > 0$), the roots of this polynomial equation are negative or have negative real parts. So, as the roots are regular functions of the delay, the steady state is locally asymptotically stable if the delay is small enough. Furthermore, for any delay there are no positive or zero solutions z of the transcendental Eq. (79).

So, if the control delay destabilizes the steady state, this will happen in an oscillatory manner, for $z(t_c) = \alpha(t_c) + i\omega(t_c)$ such that for a critical control delay $t_c = t_{c*}$ we have $z(t_{c*}) = i\omega(t_{c*})$. Substituting $z = i\omega$ in (79) we obtain:

$$\frac{P_{TH}(i\omega)}{P_c(i\omega)} = e^{-i\omega t_c}$$

Taking the moduli, squaring and reordering we obtain:

$$S_c(\mu) = \mu^2 + A_c\mu + B_c = 0$$

With $\mu = \omega^2, A = a_{0,e}^2 - c_{0,c}^2 - 2b_{0,e},\ B_c = b_{0,e}^2 - d_{0,c}^2$

So, the transcendental Eq. (79) will have a purely imaginary root $z = i\omega$ if and only if $S_c(\mu) = 0$ has a positive root $\mu_{c,*} = \omega_{c,*}^2$. If this is the case, $\omega_{c,*} = +\sqrt{\mu_{c,*}}$ will be the critical frequency.

The corresponding **critical delay** will be given by:

$$t_{c*} = \frac{1}{\omega_{c,*}}\arctan\left(\omega_{c,*}\frac{(a_{0,e}d_{d,c} - b_{0,e}c_{0,c}) + c_{0,c}\omega_{c,*}^2}{b_{0,e}d_{0,c} + (a_{0,e}c_{0,c} - d_{0,c})\omega_{c,*}^2}\right) \tag{80}$$

Now, let us remind the parameters of the neutronic-thermo-hydraulic polynomial,

$$a_{0,e} = \frac{h}{C} - \frac{|c_{00}^{FV}|G_V t_V P_*}{\Lambda_e C}$$

$$b_{0,e} = \frac{(|c_{00}^{FT}| + |c_{00}^{FV}|G_V)}{\Lambda_e C}P_*$$

and of the control related polynomial

$$c_{0,c} = \frac{|c_{00}^{FC}|G_c}{\Lambda_e}P_*$$

$$d_{0,c} = \frac{|c_{00}^{FC}|G_c}{\Lambda_e}\frac{h}{C}P_*$$

According with the general results discussed in 3.3, we must calculate first $B_c = b_{0,e}^2 - d_{0,c}^2$ to see whether it is positive or negative. We obtain, introducing a thermal static gain $G_T = \frac{1}{h}$:

$$B_c = \left(G_T^2(|c_{00}^{FT}| + |c_{00}^{FV}|G_V)^2 - |c_{00}^{FC}|^2G_c^2\right) \times \left(\frac{P_* h}{\Lambda_e C}\right)^2 \tag{81}$$

From (81) we see that $B_c < 0$ or $B_c > 0$ according to the following relations between the Doppler and void feedback coefficients and the static thermal and void gains:

$$G_T(|c_{00}^{FT}| + |c_{00}^{FV}|G_V) < |c_{00}^{FC}|G_c$$

or

$$G_T(|c_{00}^{FT}| + |c_{00}^{FV}|G_V) > |c_{00}^{FC}|G_c$$

- If $G_T(|c_{00}^{FT}| + |c_{00}^{FV}|G_V) < |c_{00}^{FC}|G_c$, for each steady state power P_* there will be a critical time delay such that when the delay is growing, and attains and overcomes this critical value, the steady state of the global mode amplitude losses its local stability in an oscillatory mode. The critical delay $t_{c*}(P_*)$ can be calculated by Eq. (80).
- On the contrary, if $G_T(|c_{00}^{FT}| + |c_{00}^{FV}|G_V) > |c_{00}^{FC}|G_c$ there will be critical delays when and only when:

$$c_{0,c}^2 + 2\left(b_{0,e} - \sqrt{b_{0,e}^2 - d_{0,c}^2}\right) > a_{0,e}^2$$

From this last inequality and from the definition of the parameters for the global mode we derive the equivalent inequality: $a_P P_*^2 + b_P P_* + c_P > 0$

$$a_P = \frac{\left[|c_{00}^{FC}|^2G_c^2 - \left(G_c - \frac{G_V|c_{00}^{FV}|\cdot t_V}{C}\right)^2\right]}{\Lambda_e^2}$$

$$b_P = \frac{2h}{\Lambda_e C}\left[G_T(|c_{00}^{FT}| + |c_{00}^{FV}|G_V)\left(1 - \sqrt{1-\sigma_c^2}\right) - \left(|c_{00}^{FC}|G_c - \frac{G_V|c_{00}^{FV}|\cdot t_V}{C}\right)\right]$$

$$c_P = -\frac{h^2}{C^2}$$

Here, by definition:

$$\sigma_c = \frac{|c_{00}^{FC}|G_c}{G_T(|c_{00}^{FT}| + G_V|c_{00}^{FV}|)}$$

is less than one. Equation

$$a_P P_*^2 + b_P P_* + c_P = 0$$

has the roots

$$P_{*1,2} = \frac{-b_P \mp \sqrt{b_P^2 + a_P|c_P|}}{2a_P}$$

If $|c_{00}^{FC}|G_c > \frac{G_V|c_{00}^{FV}|\cdot t_V}{C}$, then $a_P > 0$ and there is only one positive root

$$P'_* = \frac{-b_P + \sqrt{b_P^2 + |a_P||c_P|}}{2|a_P|}$$

From (55),

$$\operatorname{sign}\frac{d}{dt_c}\alpha_c(t_{c,*}) = \operatorname{sign}\frac{dS_c(\mu_{c,*})}{d\mu}$$

But $\frac{dS_c(\mu_{c,*})}{d\mu} = (2a_P P'_* + b_P)P'_* > 0$ and $\alpha_c(t_{c,*}) = 0$ so, when $P_* < P'_*$ the controlled reactor is stable, but when $P_* > P'_*$ there is always a critical delay t_{c*} such that if t_c the reactor grows and crosses the critical value, the reactor gets locally unstable. The analysis of the case $a_P < 0$ is similar to the analysis of the regional mode stability in the prompt-jump approximation (already done in Section 3.3.2).

Remark 8. With results obtained in Sections 3 and 4 the second objective of the present work is fulfilled.

Remark 9. Working in the framework of the effective lifetime and in the linear approximation, the evolution of the first mode amplitude follows the equation

$$\Lambda_e \frac{dP_1(t)}{dt} = (\delta\rho_{10}^F + \delta\rho_{10}^C)P_* + \rho_1^s P_1(t)$$

In the linear approximation the effect of the control loop on the regional mode weakens due to the appearance of the cross control feedback reactivity $\delta\rho_{10}^C$, so it is advisable to make a full nonlinear analysis in this case. There is an interesting property of the linear and uncoupled approximation to regional mode dynamics that can interrelate with nonlinearity and will be considered in Suárez-Ántola and Flores-Godoy (submitted for publication).

5. Mode dynamics without prompt jump approximation: a preliminary study of global mode oscillations by KBM method in the absence of automatic control effects

Let us begin with:

$$\frac{dP_0(t)}{dt} = \frac{\delta\rho_{00}}{\Lambda_0}P(t) + \frac{\beta}{\Lambda_0}\int_0^\infty D(u)(P_0(t-u) - P_0(t))du$$

Here the power corresponding to the fundamental mode $P(t) = P_* + P_0(t)$ is the total power in the core.

Let us introduce the logarithmic power:

$$x_o(t) = \ln\left(\frac{P_o(t)}{P_*}\right)$$

Then: $P = P_* e^{x_o}$, $P - P_* = P_*(e^{x_o} - 1)$. We define:

$$g(x) = e^x - 1 \tag{82}$$

The nonlinear dynamic equation for the fundamental mode can be recast as follows:

$$\frac{dx_0(t)}{dt} = \frac{\delta\rho_{00}}{\Lambda_0} + \frac{\beta}{\Lambda_0}\int_0^\infty D(u)\left(e^{x_0(t-u)-x_0(t)} - 1\right)du$$

Applying the operator $\widehat{L}_T[f(t)] = \frac{df(t)}{dt} + \frac{h}{C}f(t)$ to both members of the equation, and taking into account that

$$\widehat{L}_T[\delta\rho_{00}] \approx c_{00}^{FT}\frac{P_*}{C}g(x_0) + c_{00}^{FV}G_V\frac{P_*}{C}\int_0^\infty h_V(u)g(x_0(t-u))du$$

$$\widehat{L}_T\left[\int_0^\infty D(u)\left(e^{x_0(t-u)-x_0(t)} - 1\right)du\right] = \int_0^\infty D(u)\left(\frac{dg(x_0(t-u)-x_0(t))}{dt} + \frac{h}{C}g(x_0(t-u)-x_0(t))\right)du$$

$$g(x(t-u)-x(t)) = e^{x(t-u)-x(t)} - 1$$

Then, we find this nonlinear differential-integral equation for an oscillator in terms of the logarithmic power:

$$\frac{d^2x_0(t)}{dt^2} + \frac{h}{C}\frac{dx_0(t)}{dt} + |c_{00}^{FT}|\frac{P_*}{\Lambda_0 C}g(x_0) + |c_{00}^{FV}|G_V\frac{P_*}{\Lambda_0 C} \times \int_0^\infty h_V(u)g(x_0(t-u))du - \frac{\beta}{\Lambda_0} \times \int_0^\infty D(u)\left(\frac{dg(x_0(t-u)-x_0(t))}{dt} + \frac{h}{C}g(x_0(t-u)-x_0(t))\right)du = 0 \tag{83}$$

Now, let us introduce a thermal ω_T (prompt effect) and a void ω_V (delayed effect) frequency, by the following definitions:

$$\omega_T = \sqrt{|c_{00}^{FT}|\frac{P_*}{\Lambda_0 C}} \tag{84}$$

$$\omega_V = \sqrt{|c_{00}^{FV}|G_V\frac{P_*}{\Lambda_0 C}} \tag{85}$$

Both ω_T^2 and ω_V^2 are proportional to P_*/Λ_0 Then (83), the nonlinear oscillator equation for the fundamental mode may be recast as follows:

$$\frac{d^2x_0(t)}{dt^2} + \frac{h}{C}\frac{dx_0(t)}{dt} + \omega_T^2 g(x_0) + \omega_V^2\int_0^\infty h_V(u)g(x_0(t-u))du - \frac{\beta}{\Lambda_0}\int_0^\infty D(u)\left(\frac{dg(x_0(t-u)-x_0(t))}{dt} + \frac{h}{C}g(x_0(t-u)-x_0(t))\right)du = 0$$

Let us introduce, as explained in Appendix A, the following ansatz for the logarithmic power: $x(t) \approx \rho(t)\cos(\psi(t))$. Here $\psi(t) = \omega t + \theta(t)$ with ω a certain fast frequency to be determined and $\rho(t), \theta(t)$ slowly varying functions in comparison with the time scale $2\pi/\omega$. This being the case, it is possible to apply the KBM asymptotic method o nonlinear integral differential equation of the logarithmic power.ion.

Following the procedure outlined in Appendix A, we derive these dynamic equations:

$$\frac{d\rho}{dt} \approx \rho(\kappa_0 + \kappa_2\rho^2 + \kappa_4\rho^4 + \ldots) \tag{86}$$

$$\frac{d\psi}{dt} \approx \omega + \delta_2(\omega)\rho^2 + \delta_4(\omega)\rho^4 + \ldots \tag{87}$$

Detailed formulae for the parameters $\kappa_0, \kappa_2, \kappa_4$ and δ_2, δ_4 can be found in Appendix B. For one group of delayed neutron emitters the fast frequency verifies the equation:

$$\omega^2 \approx \omega_T^2 + \omega_V^2\int_0^\infty h_V(u)\cos(\omega u)du + \frac{\beta}{\Lambda_0}\left(\frac{h}{C} - \lambda\right)\frac{\omega^2}{\lambda^2 + \omega^2}$$

A detailed model for the delayed void feedback kernel $h_V(t)$ must be introduced now to continue the analysis. For example, in case of the well-known case given by Eq. (22)

$$\int_0^\infty h_V(u)\cos(\omega u)du = \frac{b_2 - \left(1 + \frac{b_4}{b_3}b_1\right)\omega^2}{\Delta(\omega)}$$

$$\int_0^\infty h_V(u)\sin(\omega u)du = \frac{\omega\left(b_1 + \frac{b_4}{b_3}(b_2 - \omega^2)\right)\omega^2}{\Delta(\omega)}$$

in these formulae $\Delta(\omega) = (b_2 - \omega^2)^2 + b_1^2\omega^2$. The parameters b_1, b_2, b_3 and b_4 can be expressed in terms of the residence time of steam bubbles in the channel $\tau \approx H/V_0$ and the channel height H, according to (23)–(26). But V_0 is a mean void propagation velocity in principle proportional to the coolant flow F in the fuel channels, so $\tau \propto (1/F)$. From Eqs. (84) and (85) it follows that both the thermal frequency ω_T and the void frequency ω_V are proportional to $\sqrt{P_*}$, so they both increase when the reactos steady power P_* increases. This allows us to give the parameters $\kappa_0, \kappa_2, \kappa_4$ of Eq. (86) for the amplitude as well as the parameters of Eq. (87) for the phase of the logarithmic power as analytic functions of the steady state power, coolant flow and other relevant parameters from reactors thermal–hydraulics and neutronics.

Once solved the equation for the fast frequency, it is possible to calculate the coefficients of the equation for the time variation of the amplitude of oscillation (86) and the time variation of the phase of the nonlinear oscillator (87). The sign of the coefficient κ_0 determines the local stability of the steady operation power of the reactor.

$$2\kappa_0 = -\frac{h}{C} + \frac{\omega_V^2}{\omega}\int_0^\infty h_V(u)\sin(\omega u)du - \frac{\beta}{\Lambda_0}\left[\frac{(\omega^2 + \frac{h}{C}\lambda)}{\lambda^2 + \omega^2}\right] \tag{88}$$

Remark 10. The equation $\kappa_0 = 0$ defines a hyper-surface in the **multidimensional space**of the **neutronic** and **termal-hydraulics** parameters of the BWR: this is a **stability boundary** that could be projected into the power-flow plane once certain model parameters are related to coolant flow.

In order to move faster, it is possible to simplify the mathematical model of the global mode amplitude introducing an instantaneous feedback Doppler effect with negative coefficient c_i and an retarded feedback effect of negative static coefficient c_r and time delay t_r (mainly due to voids effects), and a lumped neutron delay time t_D. Then the following formulae are obtained for the frequency ω and the coefficients in the equations for the time variation of the amplitude ρ in the ansatz for the logarithmic power

$$\Lambda_0 \cdot \omega \approx |c_r| \cdot \sin(\omega \cdot t_r) P_* - \beta \cdot \sin(\omega \cdot t_D) \tag{89}$$

$$\kappa_0(P_*, \xi) = m \cdot P_* - \beta \cdot \xi \tag{90}$$

$$8 \cdot \kappa_2(P_*, \xi) = m \cdot P_* - 2 \cdot \beta \cdot \xi^2 \tag{91}$$

$$192 \cdot \kappa_4(P_*, \xi) = m \cdot P_* - 4 \cdot \beta \cdot \xi^3 \tag{92}$$

$$m = -(|c_i| + |c_r| \cdot \cos(\omega \cdot t_r)) \tag{93}$$

$$\xi = 1 - \cos(\omega \cdot t_D) = 2 \cdot \sin^2\left(\frac{\omega \cdot t_D}{2}\right) \tag{94}$$

If $m < 0$ the steady state of the reactor is always stable.

If $m > 0$, or equivalently, if $\cos(\omega t_r) < -|c_i|/|c_r|$, there is a threshold power $P_{*,u} = \beta\xi/m$ such that $\kappa_0 = 0$. If $P_* < P_{*,u}$ the steady state is locally (linearly) stable ($\kappa_0 < 0$) and if $P_* > P_{*,u}$ it is unstable ($\kappa_0 > 0$).

When $0 < \xi < \frac{1}{2}$ then $\beta\xi > 2\beta\xi^2 > 4\beta\xi^3$ so

$\kappa_0(P_*, \xi) < 8\kappa_2(P_*, \xi) < 192\kappa_4(P_*, \xi)$

When $\frac{1}{2} < \xi < 2$ then $\beta\xi < 2\beta\xi^2 < 4\beta\xi^3$ so

$\kappa_0(P_*, \xi) > 8\kappa_2(P_*, \xi) > 192\kappa_4(P_*, \xi)$

So, fixing ξ, and varying the steady power near the threshold we have two possibilities:

- For $0 < \xi < \frac{1}{2}$, near the stability boundary, κ_0 can be negative but κ_2 can be positive as well as κ_4. So, according to the model, there is a sub-critical Poincaré–Andronov–Hopf (PAH) bifurcation in this case: when $P_* < P_{*,u}$ the locally stable steady state is surrounded by an unstable limit cycle, but when $P_* > P_{*,u}$ the limit cycle disappears and a linearly unstable steady state remains. This unstable limit cycle defines the boundary of a basin of attraction of the steady state. If a perturbation is large enough it can leave the basin of attraction of the steady state and thus the reactor would be unstable. In general the limit cycle amplitudes are given by the positive roots of

$$\kappa_0 + \kappa_2 a^2 + \kappa_4 a^4 = 0$$

are given by

$$\bar{\rho} = +\sqrt{\frac{-\kappa_2 \pm \sqrt{D}}{2\kappa_4}}$$

with $D = \kappa_2^2 - 4\kappa_0\kappa_4$.
If it is possible to neglect the influence of κ_4, the amplitude of the unstable limit cycle is given by

$$\bar{\rho}_u \approx \sqrt{\frac{-\kappa_0(P_*, \xi)}{\kappa_2(P_*, \xi)}}$$

and its frequency can be estimated by $\bar{\omega}_u(\bar{\rho}_u) \approx \omega + \delta_2(\omega)\bar{\rho}_u^2$.

- For $\frac{1}{2} < \xi < 2$, near the stability boundary, κ_0 is positive but κ_2 can be negative as well as κ_4, so the model predicts a super-critical Hopf (PAH) bifurcation in this case: when $P_* < P_{*,u}$ we have a locally stable steady-state, but when $P_* > P_{*,u}$ a stable limit cycle appears together with a linearly unstable steady. The reactor power begins to oscillate with amplitude given approximately by

$$\bar{\rho}_s \approx \sqrt{\frac{\kappa_0}{-\kappa_2}}$$

and frequency $\bar{\omega}_s(\bar{\rho}_S) \approx \omega + \delta_2(\omega)\bar{\rho}_S^2$, if in a first approximation we neglect the influence of the last term in the equation $\kappa_0 + \kappa_2\rho^2 + \kappa_4\rho^4 = 0$.

Remark 11. Now we have explored the possibility of applying asymptotic methods in the derivation of closed form analytical formulae of the stability boundaries in the space of reactor parameters (see Remark 10) as well as for amplitudes and frequencies of global and regional power oscillations (see Appendix B and formulae (89) and (94)).

Remark 12. With the results obtained in Section 5, the third objective of the present work is fulfilled.

6. Conclusions

In the following section we present some conclusions about the work done so far. The limitation of the present model and possible improvements are consider at Suárez-Ántola and Flores-Godoy (submitted for publication).

Analytical results were obtained for two kinds of approximations, the prompt jump approximation and the effective lifetime approximation. For the prompt jump approximation the stability and bifurcation were studied without automatic control variable feedback. For the effective lifetime approximation stability and bifurcation were studied with and without automatic control variables.

6.1. Analytical results from prompt-jump approximation

The global mode was uncoupled from the regional mode to simplify the study of the proposed nonlinear model. We found that the regional mode was always coupled with the global mode even in the linear approximation. If the regional mode is fully uncoupled from the global mode all the thresholds found in the linear stability and local bifurcations analysis, disappear.

The prompt jump analysis was done on the linearized versions of the global and regional mode equations.

Equations that relate the parameters of the reactor with threshold delays to instability and threshold powers to instability were found.

We worked with two different type of mathematical approaches for the interaction of the void delay or the automatic control system delay with the dynamic of the reactor. One approach produces an ordinary delay-differential equation and the other approach produces an integral–differential equation. To study the stability of the delay-differential equation we used an algebraic method that produces formulae for critical delays and threshold of powers. For the integral–differential equations the analysis was done applying the asymptotic KBM method and analytical formulae for the attenuation coefficient (positive or negative) and the frequency of oscillation of the solution.

Also we found analytical formulae for the damping coefficient and decay ratio of the global mode amplitude and the regional mode amplitude. A byproduct of this result if neutronic noise terms are included in the present model, is a deduction and a generalization of

the phenomenological approach by forced linear oscillator equations that appear in Pázsit (1995); Pázsit et al. (2010).

6.2. Analytical results from effective life-time approximation

The effective life time approximation analysis was done on a linearized version of the global and regional mode equations with feedback control in the global case and without feedback control in the regional case. A full analytical nonlinear analysis for the global mode, the uncoupled from the feedback automatic control variables was done.

From the linearized model for the global mode with the effect of the control system, conditions for a critical control delay and its impact on the stability for the global mode are presented.

For the effective lifetime three scenaria were studied. The first scenario was the linearized model for the global and regional modes without the feedback from the control system. The second scenario was a linearized model for the global mode with the effect of the control system. The third scenario was a nonlinear model for the global mode without feedback from the control system. From the first and second scenaria, the linearized models for the global and regional modes (without the feedback from the control system), formulae for the damping ration and the critical delay for a given power were obtained. The dynamics of the reactor near the stability boundary behaves as a forced linear oscillator (forced by the nonlinear and neutronic noise terms) with a damping coefficient that changes from positive to negative when the stability boundary is crossed. From the third scenario—the nonlinear model for the global mode—we found conditions such that the global mode could present a supercritial Hopf bifurcation when certain parameters reach a critical value.

6.3. Analytical formulae obtained by KBM method

For nonlinear integral–differential equation ot the global uncoupled mode without any further approximation, applying the KBM method we described the change of amplitude and phase around the steady state near the stability boundary. Analytical conditions for the existence of a subcritical and supercritical PAH bifurcations were found as well as formulae for the size of the unstable and stable limit cycles. Also the stability boundary was obtained as a well-defined function of reactor parameters. According to KBM method, it seems that with this model structure the mode amplitude does not present hard self-excitation.

Appendix A. KBM

To zero order, the KBM method (Jackson, 1989; Minorsky, 1983; Mitropolsky and Dao, 2003) begins with the ansatz $P(t) \approx a(t)\cos\psi(t)$ to solve an equation of the type $\ddot{P}(t) = \widehat{F}\left[P, \dot{P}\right]$ in a first approximation (the method allows to obtain higher order approximations, but we do not need them in this paper. Notice that $\widehat{F}\left[P, \dot{P}\right]$ in general represents an operator applied to the functions P and $\dot{P}$). The phase is given by $\psi(t) = \omega t + \theta(t)$, and the assumption is made that both $a(t)$ and $\psi(t)$ being **slowly varying functions of time** (time scales at least an order of magnitude greater than $2\pi/\omega$). During the construction of solutions of nonlinear integral–differential equations, the center frequency ω is found, after determining the equation that gives the time variation of the phase, by the condition (the point over a function of time represents the time derivative of the function): $\omega = \lim_{a\to 0}\dot{\psi}(t)$. The following restriction is imposed to the ansatz $P(t) \approx \rho(t)\cos\psi(t)$:

$$\dot{P}(t)\dot{\rho}(t)\cos\psi(t) - \rho(t)\sin\psi(t)\dot{\psi}(t) \approx -\omega\rho(t)\sin\psi(t) \tag{A.1}$$

Then:

$$\ddot{P}(t) - \omega\left(\dot{\rho}(t)\sin\psi(t) + \rho(t)\cos\psi(t)\dot{\psi}(t)\right) \approx F\left[P, \dot{P}\right] \tag{A.2}$$

Substituting $P(t)\dot{\rho}(t)\cos\psi(t)$ and $\dot{P}(t) - \omega\dot{\rho}(t)\sin\psi(t)$ in formula for $\widehat{F}\left[P, \dot{P}\right]$, and inserting the result in (A.2) we can solve simultaneouly for $\dot{\rho}(t)$ and $\dot{\psi}(t)$ form Eqs. (A.1) and (A.2) to obtain (Jackson, 1989; Minorsky, 1983; Mitropolsky and Dao, 2003):

$$\dot{\rho}(t) = A(\rho,\psi) = -\omega\rho(t)\sin\psi\cos\psi + \frac{1}{\omega}F\left[\rho(t)\cos\psi, -\omega\rho(t)\sin\psi\right]\sin\psi \tag{A.3}$$

$$\dot{\psi}(t) = \Psi(\rho,\psi) = \omega\sin^2\psi - \frac{1}{\rho\omega}F\left[\rho\cos\psi, -\omega\rho\sin\psi\right]\cos\psi \tag{A.4}$$

As the functions of $\rho(t)$ and $\psi(t) - \omega t$ vary slowly during a period $T = 2\pi/\omega$, a moving time average $\frac{1}{T}\int_t^{t+T}(\cdot)dt'$ is applied to both members of (A.3) and (A.4). The quotients

$$\frac{\rho(t+T) - \rho(t)}{T}, \quad \frac{\psi(t+T) - \psi(t)}{T}$$

are approximated by $\dot{\rho}(t)$ and $\dot{\psi}(t)$, respectively. The next step is to realize that $d\psi\omega dt$, so it is possible to make the approximation

$$\frac{1}{T}\int_t^{t+T}(\cdot)dt' \approx \frac{1}{2\pi}\int_{\psi}^{\psi+2\pi}(\cdot)d\psi'$$

Thus, we obtain, after taking into account the periodicities of the sine and cosine functions, the following equations of evolution:

$$\dot{\rho} = \frac{1}{2\pi\omega}\int_0^{2\pi}\widehat{F}\left[\rho\cos\psi, -\omega\rho\sin\psi\right]\sin\psi d\psi \tag{A.5}$$

$$\dot{\psi} = \frac{1}{2}\omega - \frac{1}{2\pi\omega\rho}\int_0^{2\pi}\widehat{F}\left[\rho\cos\psi, -\omega\rho\sin\psi\right]\cos\psi d\psi \tag{A.6}$$

Appendix B. KBM applied to integral–differential equations

Let us calculate the integrals (A.5) and (A.6) in Appendix A for the nonlinear dynamics of the global mode in terms of its logarithmic power $x(t)$.

Given the ansatz $x(t)\rho(t)\cos\psi(t)$ and taking into account the slow variation of the amplitude $\rho(t)$ and phase difference $\theta(t) = \psi - \omega t$ during a period $2\pi/\omega$, it is possible to put

$$x(t-u) \approx \rho(t)\cos\left(\psi(t) - \omega_0 u\right) \tag{B.1}$$

$$x(t-u) - x(t) \approx -\rho(t)U(\omega_0 u)\cdot\cos\left(\psi(t) + \phi(\omega_0 u)\right) \tag{B.2}$$

Here, by definition:

$$U(\omega_0 u)\cos\phi(\omega_0 u) = 1 - \cos\left(\omega_0 u\right) \tag{B.3}$$

$$U(\omega_0 u)\sin\phi(\omega_0 u) = \sin\left(\omega_0 u\right) \tag{B.4}$$

In addition, we need the following results that can be found in Abramowitz and Stegun (1965). The technique used to apply KBM to this type of integra-differential equation can be found in Suárez-Ántola (2009). Using (82)

$$g(\rho\cos\psi) = e^{\rho\cos\psi} - 1 = (I_0(\rho) - 1) + 2\sum_{n=1}^{\infty}I_n(\rho)\cos(n\psi) \tag{B.5}$$

With

$$I_n(\rho) = \left(\frac{\rho}{2}\right)^n\left(\frac{1}{n!} + \frac{\rho^2}{4(n+1)!} + \frac{\rho^4}{32(n+2)!} + \cdots\right)$$

for $n = 0, 1, 2, 3, \ldots$, then

$$\frac{2I_1(\rho)}{\rho} = 1 + \frac{\rho^2}{8} + \frac{\rho^4}{192} + \cdots \tag{B.6}$$

$$I_0(a) - I_2(a) = \frac{2}{a} I_1(a) \tag{B.7}$$

From (B.1–B.5):

$$g(\rho\cos(\psi - \omega_0 u)) = (I_0(\rho) - 1) + 2\sum_{n=1}^{\infty} I_n(\rho)\cos n(\psi - \omega_0 u) \tag{B.8}$$

$$g(-\rho U\cos(\psi + \varphi)) = (I_0(\rho U) - 1) + 2\sum_{n=1}^{\infty} (-1)^n I_n(\rho U)\cos n(\psi + \phi) \tag{B.9}$$

Taking into account (B.6)–(B.8) and (B.9) the following nonlinear **equations of evolution** may be derived by a straightforward (but lengthy) calculation of the integrals in (A.5) and (A.6) of Appendix A:

$$\frac{d\rho}{dt} = \rho(\kappa_0 + \kappa_2\rho^2 + \kappa_4\rho^4 + \ldots)$$

$$\frac{d\psi}{dt} = \omega + \delta_2(\omega)\rho^2 + \delta_4(\omega)\rho^4 + \ldots$$

By definition:

$$H_r(\omega_0) = \int_0^{\infty} h_V(u)\cos(\omega u)du,$$

$$H_i(\omega_0) = \int_0^{\infty} h_V(u)\sin(\omega u)du$$

$$D_r(\omega_0) = \int_0^{\infty} D(u)\cos(\omega u)du,$$

$$D_i(\omega_0) = \int_0^{\infty} D(u)\sin(\omega u)du$$

with

$$\omega^2 = \omega_T^2 + \omega_V^2 H_r(\omega) - \frac{\beta}{\omega\Lambda}\left(-\frac{h}{C}(1 - D_r(\omega)) + \omega D_i(\omega)\right)$$

$$2\kappa_0 = -\frac{h}{C} + \frac{\omega_V^2}{\omega} H_i(\omega) - \frac{\beta}{\omega\Lambda}\left(\frac{h}{C}(1 - D_r(\omega)) - \omega D_i(\omega)\right)$$

$$12\kappa_2 = \frac{\omega_V^2}{\omega} H_i(\omega) - \frac{4\beta}{\omega\Lambda}\left[\omega\int_0^{\infty} D(u)2\sin^4\left(\frac{\omega u}{2}\right)du + \frac{h}{C}\int_0^{\infty} D(u)\sin(\omega u)\sin^2\left(\frac{\omega u}{2}\right)du\right]$$

$$384\kappa_4 = \frac{\omega_V^2}{\omega} H_i(\omega) - \frac{16\beta}{\omega\Lambda}\left[\omega\int_0^{\infty} D(u)2\sin^6\left(\frac{\omega u}{2}\right)du + \frac{h}{C}\int_0^{\infty} D(u)\sin(\omega u)\sin^4\left(\frac{\omega u}{2}\right)du\right]$$

$$16\omega\delta_2(\omega) = \omega_T^2 + \omega_V^2 H_r(\omega) - \frac{4\beta}{\Lambda}\left(-\frac{h}{C}\int_0^{\infty} D(u)2\sin^4\left(\frac{\omega u}{2}\right)du + \omega\int_0^{\infty} D(u)\sin(\omega u)\sin^2\left(\frac{\omega u}{2}\right)du\right)$$

$$384\omega\delta_4(\omega) = \omega_T^2 + \omega_V^2 H_r(\omega) - \frac{16\beta}{\Lambda}\left[-\frac{h}{C}\int_0^{\infty} D(u)2\sin^8\left(\frac{\omega u}{2}\right)du + \omega\int_0^{\infty} D(u)\sin(\omega u)\sin^4\left(\frac{\omega u}{2}\right)du\right]$$

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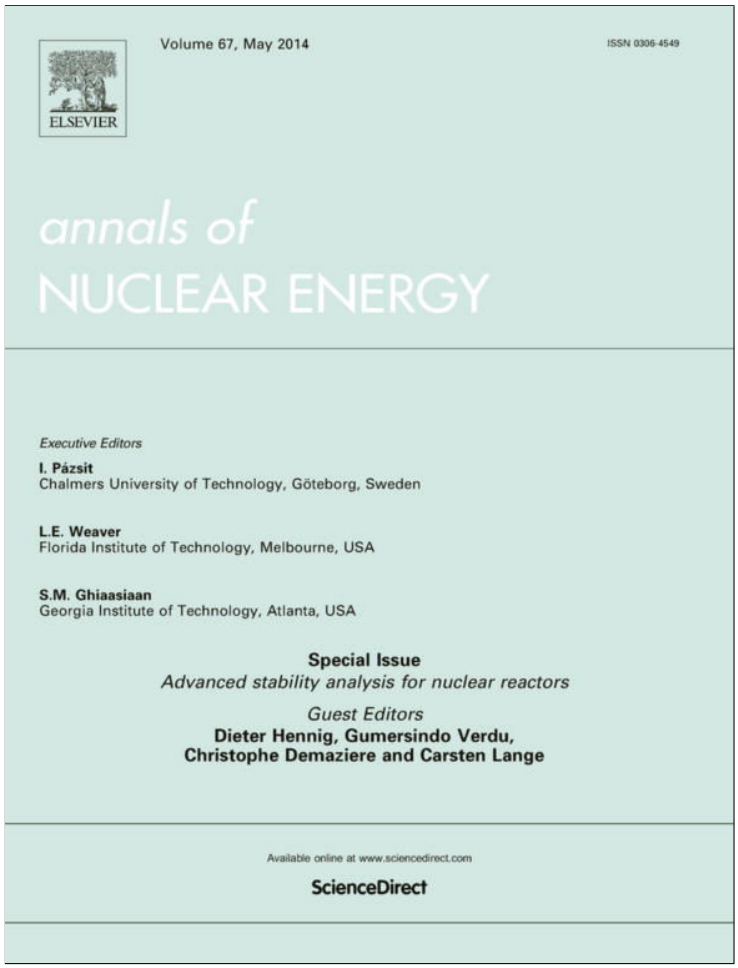

Annals of Nuclear Energy 67 (2014) 146–153

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Contribution to BWR stability analysis. Part II: Numerical approach using a reduced order model

Roberto Suárez-Ántola [a,b,*], José-Job Flores-Godoy [c]

[a] Departamento de Matemáticas, Facultad de Ingeniería y Tecnologías, Universidad Católica del Uruguay Damaso A. Larrañaga, Montevideo, Uruguay
[b] Dirección General de Secretaría, Ministerio de Industía, Energía y Minería, Montevideo, Uruguay
[c] Departamento de Física y Matemáticas, Universidad Iberoamericana, México, D.F., Mexico

ARTICLE INFO

Article history:
Received 1 November 2013
Accepted 11 November 2013
Available online 17 December 2013

Keywords:
Boiling water reactors
Stability theory
In-phase and Out-of-phase oscillations
Reduced order models
Mathematical modeling
Digital simulations

ABSTRACT

Using the reduced order model and its related analysis done in Suárez-Ántola and Flores-Godoy (submitted for publication), we study some aspects of the onset of power oscillations using numerical methods and digital simulations. From the analytical results we illustrate the usefulness of asymptotic methods to describe the change in behavior of the decay ratio and frequency of oscillations near the stability boundary in the reactor's parameter space. We study through a dynamical simulation a supercritical Hopf bifurcation in the global mode when the effect of the regional mode on the global mode is neglected. We found that the uncoupled and linearized dynamics of the regional mode is closed related with a non-normal operator. Some of the possible consequences of the non-normality are studied using digital techniques reintroducing the effect of the regional mode on the global mode. A comparison between experimental data and predictions obtained from the present reduced order model is presented.

1. Introduction

The purpose of this paper, in the framework of the reduced order model (ROM) presented in Suárez-Ántola and Flores-Godoy (submitted for publication), is threefold:

1. To illustrate the usefulness of asymptotic methods near the boundary of stability in the space of parameters of a Boiling Water Reactor (BWR).
2. To illustrate the existence of supercritical Hopf bifurcation of the global mode amplitude using the ROM.
3. To illustrate the repercussions of the existence of a non-normal operator over the stability of a highly nonlinear system like a BWR.

2. Numerical examples

For the purpose to illustrate the application of some results obtained in Suárez-Ántola and Flores-Godoy (submitted for publication) applying the Krilov–Bogoliubov–Mitropolski asymptotic method (KBM method) to the prompt jump approximations for the global and regional mode, we present some numerical examples. The values used for the parameter are shown in Table 1. The chosen values are for illustrative purposes only. Nevertheless it is possible to obtain better estimates calibrating the model against more complex and realistic ROM, e. g. Lange et al. (2011, 2012b,c). We assume that the selected parameters are constant, although it is known that when P_* changes the value of some of these parameters can change significantly (Kok, 2009).

The strategy for this section was to change parameters G_V, b_1 and P_*. The parameter G_V is the static gain of the reactivity void effect, b_1 is inversely proportional to the delay time of the void effect and P_* is the steady-state reactor power. Once we have a set of parameters for the global mode we solve (2) to find all the real-valued ω and then use them on (1) to find all possible pairs (α, ω). For the regional mode the procedure is the same but we use Eqs. (3) and (4).

For the global mode amplitude we have:

$$\alpha = \frac{1}{2}\left\{-\left(\frac{h}{C}+\frac{|c_{00}^{FT}|}{\beta C}P_*\right)-\left(\frac{|c_{00}^{FV}|G_V}{\beta C}P_*\right)H_r(\omega)+\left(\lambda\frac{|c_{00}^{FV}|G_V}{\beta C}P_*\right)\frac{H_i(\omega)}{\omega}\right\} \tag{1}$$

$$\omega^2 = \left(\lambda|c_{00}^{FT}|\frac{P_*}{\beta C}\right)+\left(\lambda\frac{|c_{00}^{FV}|G_V}{\beta C}P\right)H_r(\omega) + \left(\frac{|c_{00}^{FV}|G_V}{\beta C}P_*\right)\omega H_i(\omega) \tag{2}$$

* Corresponding author at: Departamento de Matemáticas, Facultad de Ingeniería y Tecnologías, Universidad Católica del Uruguay Damaso A. Larrañaga, Montevideo, Uruguay.
E-mail addresses: rsuarez@ucu.edu.uy (R. Suárez-Ántola), job.flores@ibero.mx (J.-J. Flores-Godoy).

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http://dx.doi.org/10.1016/j.anucene.2013.11.028

Table 1
Numerical values of selected parameters.

	Estimated value	Units
h	0.0092	J K^{-1} s^{-1}
C	0.03994	J K^{-1}
λ	0.08	s^{-1}
Λ_0	4e − 5	s
Λ_1	4e − 5	s
β	0.0056	
ρ_1^s	−0.005	
c_{00}^{FT}	−1e − 5	K^{-1}
c_{00}^{FV}	−38e − 5	
c_{10}^{FT}	−0.01e − 5	K^{-1}
c_{10}^{FV}	−0.01e − 5	
ω_*	6.28	(rad) s^{-1}

For the regional mode amplitude:

$$\alpha = \frac{1}{2}\left\{-\left(\frac{h}{C}+\frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)}+\frac{2|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\right)-\left(\frac{2|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)H_r(\omega) + \left(\frac{2\lambda|c_{10}^{FT}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)\frac{H_i(\omega)}{\omega}\right\} \tag{3}$$

$$\omega^2 = \left(\frac{h}{C}\frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)}+\frac{2\lambda|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\right)+\left(\frac{2\lambda|c_{10}^{FT}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)H_r(\omega) + \left(\frac{2|c_{10}^{FV}|G_V P_*}{(\beta+|\rho_1^s|)C}\right)\omega H_i(\omega) \tag{4}$$

With the chosen parameters we can identify three qualitative different situations.

Remark 1. The decay ratio

$$\mathrm{DR} = e^{-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}, \quad \zeta = \frac{\alpha}{\omega} \tag{5}$$

can be obtained from the previous equations, for global mode Eqs. (1) and (2) and for the regional mode Eqs. (3) and (4). When α changes sign from positive to negative the decay ratio increases from some number smaller to one to some number larger than one.

2.1. Global and regional modes always stable

In this case when b_1 is sufficiently large an G_V sufficiently small the system is stable for any value of P_* greater than zero for both modes as can be seen in Fig. 1 panes (a) and (b) since $\alpha < 0$ for all P_*. On panes (c) and (d) from Fig. 1 we can see that for both modes the exponent real part of ansatz is negative hence the associate solution will converge to zero.

2.2. Global mode instable and regional mode stable

In this case the condition

$$|c_{10}^{FT}| < |c_{10}^{FV}|G_V \tag{6}$$

is satisfied, therefore for a large enough P_* the equilibrium for the global region becomes unstable as can be seen in Fig. 2 pane (a), in this case the instability results around $P_* \approx 1$. In the interval of P_* between 1 and 3 the solution could have three frequencies. On pane (c) of Fig. 2 the variation of the root-locus can be seen as it evolves when P_* changes. Starting around (0, 0) in deep blue, the color changes to red; when P_* goes beyond the value of 1 it can be seen the appearance of four roots with negative real part that disappear once $P_* > 3$. The regional mode is stable for all P_*.

2.3. Regional mode instable

For this case, in order for the regional mode to become unstable it is necessary for the global mode to be unstable also. The instability for the regional mode is when $P_* \approx 1000$ as can be seen in Fig. 3.

3. Nonlinear dynamic simulations

Based on the information obtained from Section 2 we chose some parameter values to illustrate some of the situations described in Suárez-Ántola and Flores-Godoy (submitted for publication). The digital simulations of the nonlinear ROM was

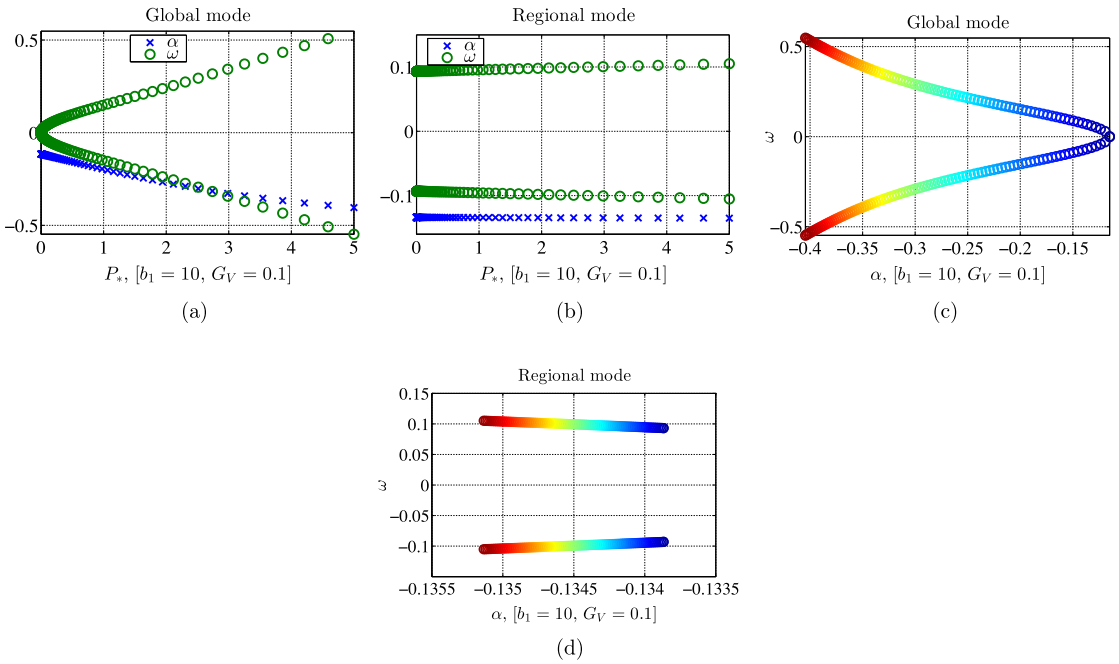


Fig. 1. Global and regional modes always stable. Pane (a) global mode, (b) regional mode, α, ω vs. P_*. Pane (c) global mode, (d) regional mode, ω vs. α.

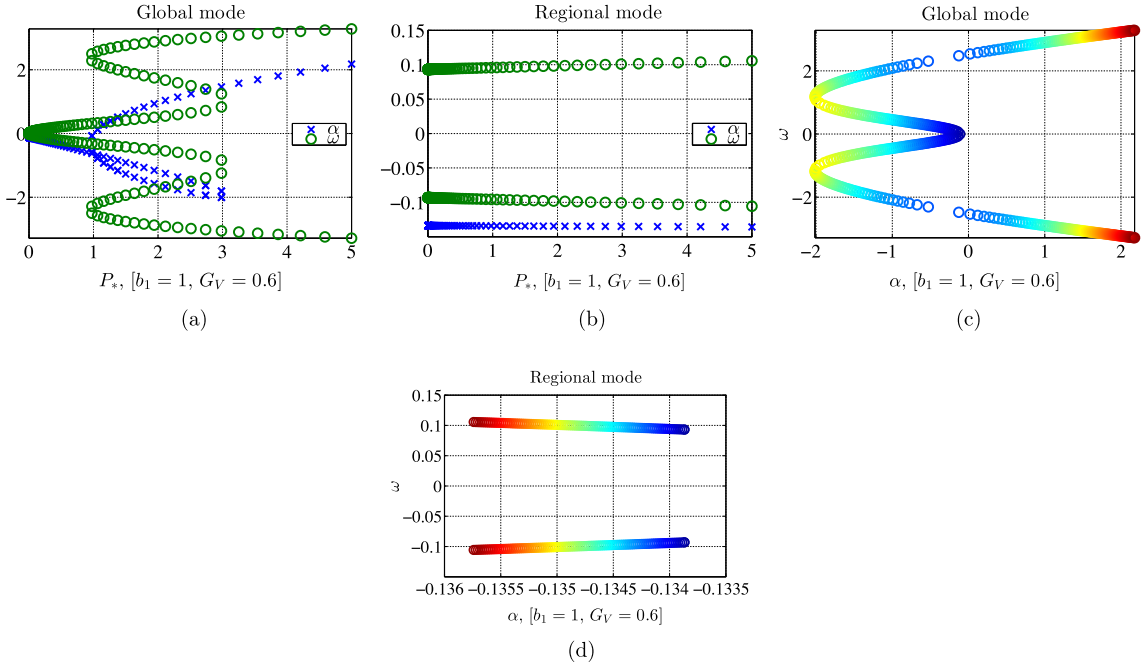


Fig. 2. Global mode instable and regional mode stable. Pane (a) global mode, (b) regional mode, α, ω vs. P_*. Pane (c) global mode, (d) regional mode, ω vs. α.

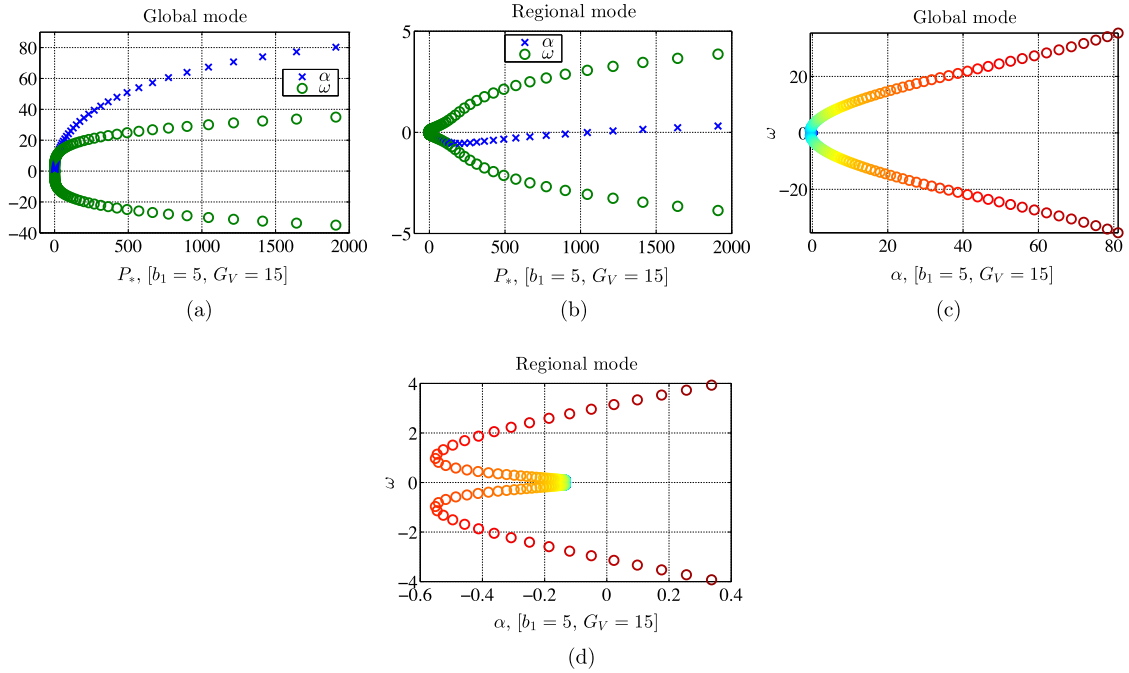


Fig. 3. Regional mode instable. Pane (a) global mode, (b) regional mode, α, ω vs. P_*. Pane (c) global mode, (d) regional mode, ω vs. α.

done with the following equations that neglect the effect of the regional mode on the global mode (see Suárez-Ántola and Flores-Godoy, submitted for publication):

$$\Lambda_0 \frac{dP_0(t)}{dt} = (\delta\rho_{00} - \beta)(P_* + P_0(t)) + \Lambda_0 \lambda C_0 \tag{7}$$

$$\frac{d}{dt} C_0 = \frac{\beta}{\Lambda_0}(P_* + P_0(t)) - \lambda C_0 \tag{8}$$

$$\Lambda_1 \frac{dP_1(t)}{dt} = \delta\rho_{10}(P_* + P_0(t)) + (\rho_1^s - \beta + \delta\rho_{11})P_1(t) + \Lambda_1 \lambda C_1 \tag{9}$$

$$\frac{d}{dt} C_1 = \frac{\beta}{\Lambda_1} P_1(t) - \lambda C_1 \tag{10}$$

$$C \frac{d\delta T_0}{dt} = (P - P_*) - h\delta T_0 \tag{11}$$

$$C \frac{d(\Delta T)}{dt} = 2P_1 - h\Delta T \tag{12}$$

$$\frac{d^2\delta\alpha}{dt^2} + b_1 \frac{d\delta\alpha}{dt} + b_2 \delta\alpha = b_3 \delta T_0 + b_4 \frac{d\delta T_0}{dt} \tag{13}$$

$$\frac{d^2\Delta\alpha}{dt^2} + b_1 \frac{d\Delta\alpha}{dt} + b_2 \Delta\alpha = b_3 \Delta T_0 + b_4 \frac{d\Delta T_0}{dt} \tag{14}$$

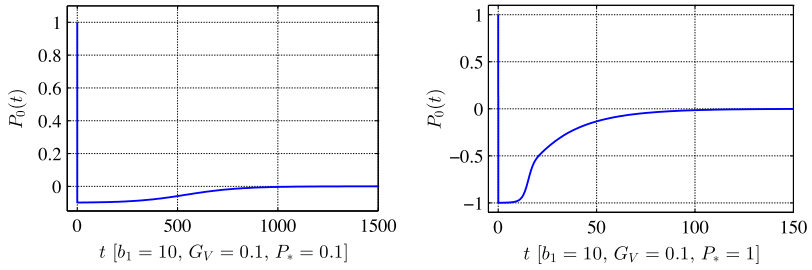


Fig. 4. Global mode stable, (a) $P_* = 0.1$, (b) $P_* = 1$.

$$\begin{aligned}\delta\rho_{00} &\approx c_{00}^{FT}\delta T_0 + c_{00}^{FV}\delta\alpha\\ \delta\rho_{11} &\approx c_{11}^{FT}\delta T_0 + c_{11}^{FV}\delta\alpha\\ \delta\rho_{10} &\approx c_{01}^{FT}\Delta T + c_{01}^{FV}\Delta\alpha\end{aligned} \tag{15}$$

The simulation were made using Matlab the values for the parameters used are presented in Table 1. In all cases the regional mode has a stable behavior with a very fast convergence towards the equilibrium (time scale around 10^{-3} s), therefore none of their graphs are shown.

3.1. Global and regional modes always stable

In Fig. 4 we can see the evolution of P_0 with different values of P_*. As we can see, the response is stable but the time constant depending on the parameters might be very slow. This result is expected from the analysis of Section 3 of Suárez-Ántola and Flores-Godoy (submitted for publication).

3.2. Global mode instable

In Fig. 5 we can see the evolution of P_0 with different values of P_*. As we can see, the response changes from stable to oscillatory accordingly with the value of P_*. In pane (b) the value of P_* is marginally larger than the boundary-value between stable/unstable behavior in P_0. In pane (c) the behavior still is periodic but the behavior is far from a simple oscillation. For this value of P_* the result found in Section 3 of Suárez-Ántola and Flores-Godoy (submitted for publication) is not longer accurate. Since P_* is far from 0 the frequency of oscillation is distorted, the double frequency shown in pane (c) cannot be accounted by the stability analysis near the boundary.

4. The equation for the regional mode in the prompt-jump approximation: linearization, uncoupling and non-normal stability

Let us consider the linearized equation for the regional mode amplitude given in Suárez-Ántola and Flores-Godoy (submitted for publication):

Table 2
Numerical values of selected parameters.

	Estimated Value	Units
Λ_0	4e − 5	s^{-1}
Λ_1	4e − 5	s^{-1}
h	0.0092	J K^{-1} s^{-1}
C	0.03994	J K^{-1}
λ	0.08	s^{-1}
β	0.0056	
ρ_1^s	−0.0001	
c_{00}^{FT}	−1e − 5	K^{-1}
c_{00}^{FV}	−38e − 5	
c_{11}^{FT}	−1.5680e − 5	K^{-1}
c_{11}^{FV}	−57.6e − 5	
c_{01}^{FT}	−0.2e − 7	K^{-1}
c_{01}^{FV}	−0.2e − 7	
c_{10}^{FT}	−1e − 7	K^{-1}
c_{10}^{FV}	−1e − 7	
b_1	1	s^{-1}
b_2	6.28	(rad) s^{-1}
G_V	0.495	

$$\frac{d^2P_1(t)}{dt^2} + a_1\frac{dP_1(t)}{dt} + b_1P_1(t) + c_1\mathcal{H}_V\left(\frac{dP_1(t)}{dt}\right) + d_1\mathcal{H}_V(P_1(t)) = 0 \tag{16}$$

with

$$\begin{aligned}a_1 &= \frac{h}{C} + \frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\\ b_1 &= \frac{h}{C}\frac{\lambda|\rho_1^s|}{(\beta+|\rho_1^s|)} + \frac{2\lambda|c_{10}^{FT}|P_*}{(\beta+|\rho_1^s|)C}\\ c_1 &= \frac{2|c_{10}^{FV}|G_VP_*}{(\beta+|\rho_1^s|)C} \qquad d_1 = \frac{2\lambda|c_{10}^{FV}|G_VP_*}{(\beta+|\rho_1^s|)C}\end{aligned} \tag{17}$$

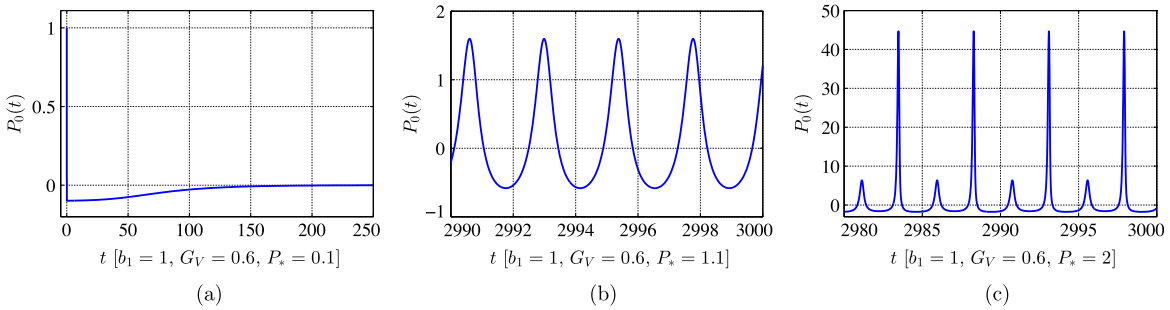
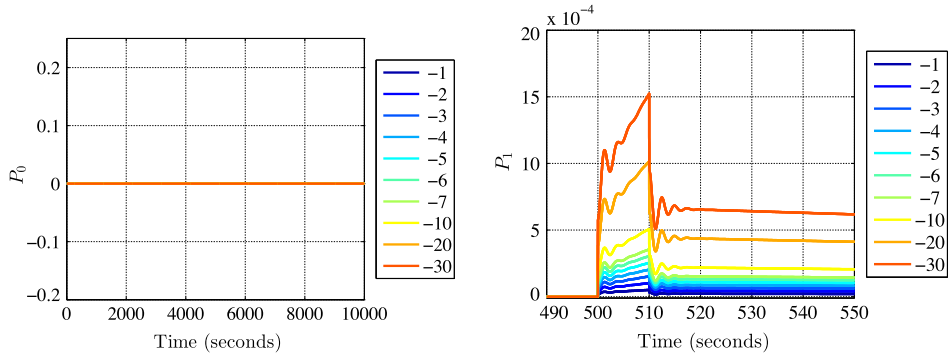
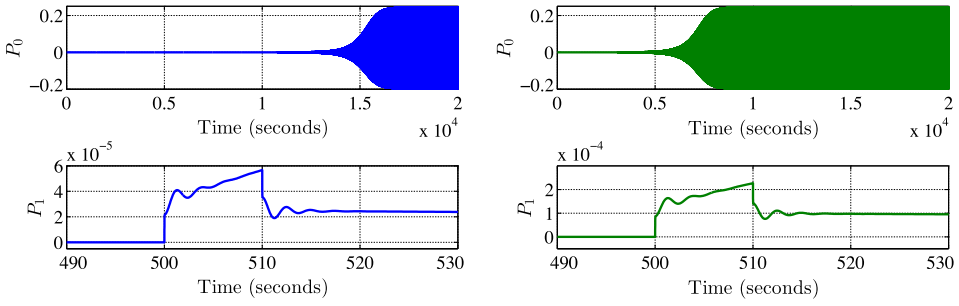


Fig. 5. Global mode stable (a) $P_* = 0.1$, oscillatory (b) $P_* = 1.1$ and (c) $P_* = 2$.

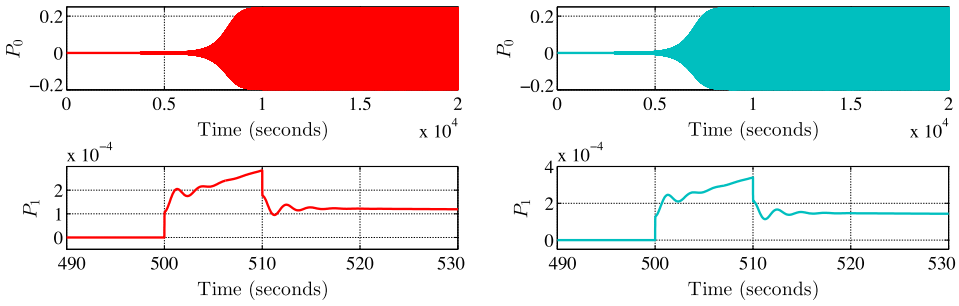


(a) Global mode amplitude. (b) Regional mode amplitude.

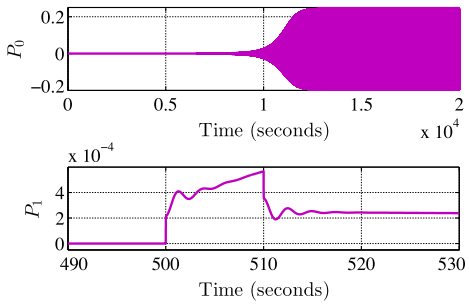
Fig. 6. Reactor steady state stable under small perturbations with $P_* = 1.1$.



(a) Perturbation amplitude −1. (b) Perturbation amplitude −4.

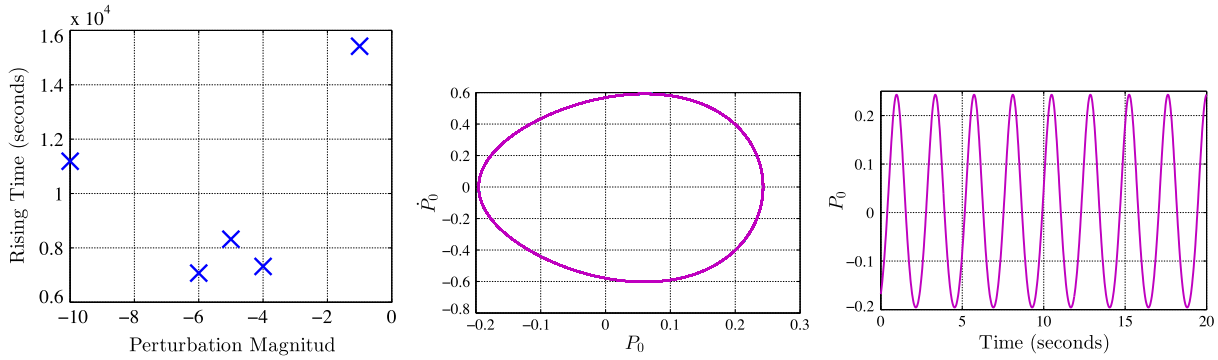


(c) Perturbation amplitude −5. (d) Perturbation amplitude −6.



(e) Perturbation amplitude −10.

Fig. 7. Reactor steady state oscillatory under small perturbations with $P_* = 1.229$.



(a) Time lag to limit cycle (b) Closed orbit of limit cycle (c) Time series of limit cycle

Fig. 8. Limit cycle properties. (a) Time lag to limit cycle. (b) Closed orbit of limit cycle. (c) Time series of limit cycle.

We uncouple the global mode from the regional one, i.e., $c_{10}^{FT} = c_{10}^{FV} = 0$.

With the following definitions $P_1(t) = u_1, a = \frac{h}{C}$ and $c = \frac{\lambda|\rho_1^s|}{\beta + |\rho_1^s|}$, introducing a new variable defined by $u_2 = \frac{du_1}{dt} + au_1$, we obtain

$$\frac{d}{dt}\vec{u} = A\vec{u}, \quad A = \begin{bmatrix} -a & 1 \\ 0 & -c \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{18}$$

as can be seen in Suárez-Ántola and Flores-Godoy (2013). The A matrix is a non-normal operator, so that if both a and c are small enough relative to 1, we can expect a significant linear growth for certain solutions (Schmid, 2007). The solution of (18), without forcing terms due to mode coupling and **non-linear** terms, with the following initial conditions: (a) the reactor is in its steady state for every $t < 0$, i.e., $P_1(t) = 0$ and $\dot{P}_1(t) = 0$ for every $t < 0$, and (b) a sudden perturbation $\dot{P}_1(0^+)$ in the velocity of the regional mode P_1 appears at $t = 0$, is given by:

$$P_1(t) = \frac{\dot{P}_1(0^+)}{a - c}\left(e^{-ct} - e^{-at}\right), \; a \neq c \tag{19}$$

If the time evolution of the regional mode amplitude $P_1(t)$ follows (19), then $|P_1(t)|$ reaches a maximum value at the instant:

$$t_* = \frac{1}{a - c}\log_e\left(\frac{a}{c}\right)$$

This maximum is given by

$$P_1(t_*) = \frac{\dot{P}_1(0^+)}{a - c}\left[\left(\frac{c}{a}\right)^{\frac{c}{a-c}} - \left(\frac{c}{a}\right)^{\frac{a}{a-c}}\right]$$

As an example, let us suppose that $a = \epsilon$ and $c = 2\epsilon$. In this case $P_1(t_*) = \left[\dot{P}_1(0^+)\right]/(4\epsilon)$, so for a given $\dot{P}_1(0^+)$, if ϵ tends to zero $|P_1(0^+)|$ tends to infinite.

A consequence of this, in the non-linear regime, we could expect a meeting of non-normality with the nonlinearity. The importance of this event is well known in fluid mechanics (Schmid, 2007):

- It is possible that the regional mode amplitude could increases from zero (its rest value) in the linear domain so much that the nonlinear regime is attained and takes control of the dynamics.
- Depending of the effect of the nonlinear terms of the coupled equations for the regional and the global mode, the region of attraction of the steady state could shrink as the parameter a and c decrease.

4.1. Partial numerical simulation with the complete model

We chose a set of parameter values to exemplify stability issues related with the non-normal behavior of the regional mode seen in Tabel 2. In this case the effect on the global mode of the coupling with the regional mode is now taken into account.

The simulations started at the steady state of the reactor and lasted 20,000 s. The simulations were made using Simulink from MATLAB R2012a using the solver ode15s with a relative tolerance of $1e-10$. We perturbed the reactor by a pulse of temperature difference in the reactor halves. The pulse width is of 10 s and the amplitude of the pulse was varied between $-1\,^\circ$C and $-30\,^\circ$C. Two case were studied depending on the value of the steady state power of the reactor P_*.

4.1.1. Case I, $P_* = 1.1$

Observe that in Fig. 6(a) the global mode remains at its steady state regardless of the perturbation used. Observed that the perturbation excites the regional mode only and the response of the regional mode is very small (see Fig. 6(b)). From Fig. 6(b) we can observed that there are three different time scales in the response of the amplitude of the regional mode; the fastest time constant is due to the contribution of Λ_0. The second fastest corresponds to the prompt-jump approximation. The slowest time constant correspond to an exponential decay toward the steady state of the regional amplitude. The slowest time constant is the same for all the perturbations.

4.1.2. Case 2, $P_* = 1.229$

In this case we found three different scenarios depending on the magnitude of the temperature perturbation. When the amplitude of the perturbation was $-3, -20, -30$ the solution of the global mode after a very small intimal perturbation returns to its steady state. When the amplitude of the perturbation was $-2, -7$ the solution for global mode presents an initial small perturbation and converges to a small limit cycle. The amplitude of this limit cycle is large enough to not be considered numerical noise. However for practical purposes this oscillation can be neglected. When the amplitude of the perturbation was $-1, -4, -5, -6, -10$ a significant limit cycle is excited in the global mode amplitude and all the cases the regional mode converges back to its steady state as can be seen in Fig. 7. In Fig. 8(a) we show the time lag for the limit cycle to initiate as a function of the amplitude of the asymmetric perturbation in the core temperature. Observe that the time lag is not monotonically behaved as a function of the perturbation. In Fig. 8(b) we can see the superposition of all established

solutions: the superposition produces a unique closed orbit. In Fig. 8(c) an example of the time series of the oscillation.

5. Conclusions and discussion

5.1. Numerical results

The digital simulation was made using the full nonlinear model.

In Suárez-Ántola and Flores-Godoy (submitted for publication), we obtained analytical results for the decay ration and the frequency of oscillation near the stability boundary modeling the void delay in two different was. One modeling was using delay-differential equations and the other using integral–differential equations. What we found through the numerical study is that the stability condition found for the delay-differential equations and the integral–differential equations are compatible in the sense that when one condition describes stability the other condition also describes stability, the same is true for the unstable case.

With the full nonlinear model we produce a digital simulation for a chosen sets of reactor parameters to illustrate different behaviors found through the analytical results. The digital simulations are not exhaustive but, the shape and size of oscillation (when it happens) and the relaxation to steady state agree fairly closely with the theoretical and reported results (March-Leuba (1992); Pázsit, 1995; Maqua et al., 2002; Wehle et al., 2008; Pázsit et al., 2010).

Due to the non-normality of the uncoupled and linearized evolution operator, the threshold for hard self-excitation could be so low that in practice it would be hard to distinguish from a true soft self-excitation. This type of behavior has been already observed in fluid mechanics Schmid (2007). We found that for very mild perturbations in the temperature difference between the two reactor halves it is possible for the regional mode to excite into an oscillatory regime the global mode. Even though the magnitude of the regional mode response to the perturbation is very small, given enough time, the global mode oscillates. This happens for the steady state reactor power beyond a threshold. Also, there is evidence that there exist according to the model a very small oscillation excited for certain perturbation amplitudes and for other amplitudes the global mode returns to the steady state for the duration of the simulation.

5.2. Discussion

Early natural circulation BWR showed oscillatory instabilities at relatively high powers, and inability to follow demand variations. Forced circulation was included to improve both stability and load following capacity. The inclusion of a recirculation loop with adjustable flow and a regulation of steam flow from the reactor to hold system pressure fixed, allowed control of reactor power output variations in a relatively short time scale of changing external demands.

One possible source of instability that was encountered since the first BWR designs is the control system itself. A classical piecewise linear approach to control design, when is applied to a complex nonlinear thermal–hydraulic machine, can succeed in guaranteeing stable operation in a neighborhood of the nominal flow and nominal power, but perhaps it could fail other operating points in the power-flow map. Besides, there are controller gain settings and robustness issues due to the regular wear and tear of the components that could end in unstable operation. In any case, control instability as a rule gives rise to very low frequency (hundredths of Hz) and low amplitude (few percent) oscillations in thermo-hydraulic variables. If left undetected, these oscillations produce an inconvenient increase in control actuators wear. The new methods of control design, implementation and maintenance improved the performance of BWR control systems and diminished the importance of this kind of instability. However the present ROM introduces the automatic control effects (based on control rods only) in the framework of the model in a systematic way. As shown Suárez-Ántola and Flores-Godoy (submitted for publication), when the effective life-time approximation is done we obtain two forced (by nonlinearities) harmonic damped oscillator equations for the global and regional modes. There is a stability boundary related with the parameters of the automatic control loop and depending on their value it is possible to find conditions of unstable behavior.

The maturation process of light water reactors technologies for nuclear reactors of generations II and III shifted the emphasis from the design and construction of new reactor cores to operational considerations related with electric load, refueling and maintenance requirements. Nevertheless, one of the initial goals of nuclear power systems design and operation remains of practical importance, in spite of this change of emphasis: to impose restrictions so that the possible states of certain critical subsystems, during steady-state operation and during transients, always remain suitably bounded.

Due to power increase, which is the current trend, stability became again a limit design concern. In particular, the introduction of high efficiency fuels enables operation at higher power densities resulting in higher gain void feedback reactivities. This and other effects of the improved designs, tend to lower the stability of nuclear power plants (NPP) with BWR (Glasstone and Sesonske, 1994; Kok, 2009) and continue to pose interesting problems, mainly related with the out-of-phase oscillations.

The present ROM incorporates the possibility to include the combined effect of neutron physics, thermal considerations, and automatic control based on absorbent rods in the process of calculation of the reactivity coefficients (thermal, void and control rod) during the life time of the reactor in a systematic way.

In the following section we present some limitation on one hand and suggest some improvements on the other hand, of the model in its current state of development.

5.3. Limitations of the ROM in its present state and future work

The linear nodal-heat transfer model, based on an average fuel temperature, introduces only two parameters (a heat capacity due mainly to fuel, and a Newtonian heat transfer coefficient) that summarize the thermal processes of interest and can be considered as constants in the neighborhood of a given steady power. Working with a single average fuel temperature, in the framework of a linear model, ignoring possible coolant temperature variations and well known nonlinearities as well as the significant temperatures differences between the central and peripheral parts of a fuel pin, is of course a major simplification. The Doppler Effect and heat transfer process are over-simplified in this approach Rozon (1998).

The addition of a prompt feedback coefficient to relate reactivity to power, directly and without a significant delay, as suggested at the end of Section 3 of Suárez-Ántola and Flores-Godoy (submitted for publication), could be a way to improve the model.

The boundary between liquid and steam, with its height variations is not considered explicitly in this model. The indirect loop with its significant hydraulics is not considered, also.

To study the oscillations during a length of time of an hour or more, it would be necessary to include additional effects such as poison dynamics and heat produced by delayed neutron emitters amongst others.

It seems that in its present form, the model used in this work allows a fairly complete analytical description of local bifurcations, including soft self-excitation of nonlinear power oscillations.

However, the analysis and digital simulations done so far suggests that the linearity in the relation between the direct and cross feedback reactivities and the reactor state variables, and the linearity in the relation of this variables with global and regional mode amplitudes, as appears in the model structure, does not allow the description of global bifurcations. The possibility to explore the introduction of nonlinearities seems worthwhile. Perhaps, an additional momentum variable could be added to the present model, in order to approximately take into account some indirect loop effects related with BWR stability issues, like the hard self-excitation found by Lange et al. (2012a) with TUD–Dresden ROM.

As a first approximation we study the existence of a supercritical Hopf bifurcation in the global mode uncoupled from the regional mode. This was done considering the strength of the effect of the regional mode on the global mode is small compare to the strength of the global mode on the regional mode.

As an improvement when the coupling of the regional mode on the global mode is reintroduced, we found that the amplitude of the global mode could be unstabilized associated with very small perturbations produced in the regional mode. However the other effects of this coupling in the framework of the present ROM remains to be studied.

For future work a suitable calibration of the parameters is needed in order to reproduce a realistic reactor dynamics. To further improve the span of this model, a first step would be to include a thermal hydraulic indirected loop to be able to study less idealized regional mode oscillations. It would also be necessary to include additional effects such as poison dynamics and heat produced by delayed neutron emitters amongst others in order to be able to extend the length of time in which the model predicts a working reactor behavior.

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